LARGE DEVIATIONS FOR SPECTRAL MEASURES OF SOME SPIKED MATRICES

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Abstract. We prove large deviations principles for spectral measures of perturbed (or spiked) matrix models in the direction of an eigenvector of the perturbation. In each model under study, we provide two approaches, one of which relying on large deviations principle of unperturbed models derived in the previous work "Sum rules via large deviations" (Gamboa-Nagel-Rouault, *JFA* [13] 2016).

1. Introduction

Beside the empirical spectral distribution of an $n \times n$ random matrix M_n

$$\mu_u^{(n)} = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k},$$

whose asymptotical behavior is widely known, there has been a growing interest in the study of the so-called spectral measures. For any fixed unit vector $e^{(n)} \in \mathbb{C}^n$, the spectral measure associated to the pair $(M_n, e^{(n)})$ is the measure $\mu_{\mathtt{w}}^{(n)}$ defined by

$$\langle e^{(n)}, (M_n - z)^{-1} e^{(n)} \rangle = \int_{\mathbb{R}} \frac{d\mu_{\mathtt{w}}^{(n)}(x)}{x - z} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}$$

if M_n is self-adjoint or

$$\langle e^{(n)}, \frac{M_n + z}{M_n - z} e^{(n)} \rangle = \int_0^{2\pi} \frac{e^{\mathrm{i}\theta} + z}{e^{\mathrm{i}\theta} - z} d\mu_{\mathtt{w}}^{(n)}(\theta) \quad \text{for all } z : |z| \neq 1,$$

if M_n is unitary. In turns out that the spectral measure is a weighted version of the empirical spectral distribution:

$$\mu_{\mathtt{w}}^{(n)} = \sum_{k=1}^{n} \mathtt{w}_k \delta_{\lambda_k},$$

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where $\mathtt{w}_k = |\langle \phi_k, e^{(n)} \rangle|^2$, with ϕ_k a unit eigenvector associated to the eigenvalue λ_k. It was studied under the name *eigenvector empirical spectral distribution* in [34], in the context of unperturbed random covariance matrices.

In a series of papers [18, 13, 14, 15, 16] Gamboa et al. studied the random spectral measure $\mu_{\mathtt{w}}^{(n)}$ of a pair $(M_n, e^{(n)})$ where M_n is a random $n \times n$ matrix self-adjoint or unitary, whose distribution is invariant by conjugation, and $e^{(n)}$ is a fixed vector of $\mathbb{C}^n$. The authors proved that with convenient assumptions on the potential, the family $(\mu_{\mathtt{w}}^{(n)})_{n\geq 1}$ satisfies a large deviations principle at scale n with a good rate function consisting of two parts. The first part is the Kullback entropy of the equilibrium measure μ_V with respect to the absolute continuous part of the argument measure. The second part corresponds to the contribution of the outliers of the argument measure, namely of the eigenvalues that belong to the complement of the support of μ_V. Besides, when the spectral measure is encoded by the Jacobi recursion coefficients (or the Verblunsky coefficients in the unitary case), the rate function admits another expression in term of these coefficients, which is a simple functional in most of the classical cases. The identification of the two expressions of the rate functions leads to the so called *sum rules*.

The simplest Hermitian invariant models are the well known Gaussian Unitary Ensemble GUE(n) and Laguerre Unitary Ensemble LUE$_{n\tau}(n)$, whose equilibrium measures are respectively given by the semi-circle law (SC) and the Marchenko-Pastur law (MP$_\tau$). In the unitary world, the simplest model is of course the CUE(n) which corresponds to the Haar measure on the unitary group. The first non-trivial models are provided by the Gross-Witten measures which form a family of probability measures on the unitary group, absolutely continuous with respect to the CUE(n).

In this paper, we are interested in the large deviations of the spectral measures of rank-one perturbations of the classical aforementioned models. More precisely, we will consider additive perturbations of the GUE(n), multiplicative perturbations of the LUE$_{n\tau}(n)$ and multiplicative perturbation of the Gross-Witten measures. For a survey of the literature on these so-called *spiked* models, we refer the reader to [9]. Let us mention that, at the level of large deviations, the extreme eigenvalues have been studied in [4], and the pair (extreme eigenvalue, weight) has been recently considered in [5]. In the present work, we establish large deviations principles for the sequences of spectral measures associated to the pairs $(M_n, e^{(n)})$, in case where the reference vector $e^{(n)}$ is colinear to the eigenvector of the perturbation. The corresponding good rate functions are simple perturbations of the good rate functions of the undeformed models and we refer the reader to Theorems 5.1, 5.2 and 5.3 for precise statements.

In order to derive these large deviations principles, we propose two approaches, each based on the already known LDP for classical models, and shedding different lights on the problem. The first one uses that the distributions of the spectral measures of the deformed models are tilted versions of the distributions of the spectral measures of the undeformed ones. The second approach relies on the computations of the Jacobi (resp. Verblunsky) parameters of the deformed models.

Of course, the unique minimizers of the rate functions corresponds to the limiting spectral measures of the considered models. In particular, we recover the expressions of the limiting spectral measures associated to the perturbations of the GUE(n) and the LUE$_{n\tau}(n)$, which belong to the class of *free Meixner laws*. In the Gaussian setting, this was first observed in [25]. In the general case, this is a consequence of the local laws [24, 23], as observed in [26]. A byproduct of our considerations also yields a characterization of the limiting measures as the unique minimizers of the rate functions of the unperturbed models, under a constraint on the mean.

In a last part, we propose two generalizations of our considerations. The first one is concerned with perturbations of general invariant models, while the second one deals with matricial versions of the spectral measures.

In all the sum rules considered, the Kullback-Leibler divergence or relative entropy between two probability measures μ and ν plays a major role. When the probability space is $\mathbb{R}$ endowed with its Borel σ-field, it is defined by

$$\mathcal{K}(\mu \mid \nu) = \begin{cases} \displaystyle\int_{\mathbb{R}} \log \frac{d\mu}{d\nu}\, d\mu & \text{if } \mu \text{ is absolutely continuous with respect to } \nu, \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Usually, ν is the reference measure. Here the spectral side will involve the reversed Kullback-Leibler divergence, where μ is the reference measure and ν is the argument.

The outline of the paper is as follows. In Section 2, we present our three random models and the main notations. Section 4 gives the encoding of the spectral measures by Jacobi parameters in the real case and Verblunsky parameters in the complex case. In Section 4, we recall the results obtained by the second author of this paper with Gamboa and Nagel about large deviations and the sum rules. Section 5 contains our results, which are stated in Theorems 5.1, 5.2 and 5.3. In Section 6, we present some generalizations of these results. Finally, in an appendix we present a technical lemma and a short panorama of measures found in the different limits, which simplifies some computations along the paper.

2. Notations

In this article, we are going to consider perturbed versions of three classical models of random matrices whose definitions are recalled here. The two first models have real eigenvalues and correspond to the Hermite and the Laguerre ensembles. The third model will have its eigenvalues on $\mathbb{T} := \{z \in \mathbb{C},\, |z| = 1\}$, and corresponds to the so-called Gross-Witten measure, which is absolutely continuous with respect to the Haar measure on $\mathbb{U}(n)$.

The Hermite ensemble. For all $n \geq 1$, the Gaussian Unitary Ensemble GUE(n), or Hermite ensemble, is a probability distribution on Hermitian matrices of size $n \times n$, whose density is proportional to $\exp\left(-\frac{1}{2}\mathrm{Tr}(MM^\star)\right)$. The rescaled matrix $X = \frac{1}{\sqrt{n}} M$ has law:

$$\mathbb{P}_0^{(n)}(dX) := \frac{1}{\mathcal{Z}_0^{(n)}} \exp\left(-\frac{n}{2}\mathrm{Tr}(XX^\star)\right) dX. \tag{2.1}$$

The equilibrium measure of this ensemble, i.e. the limit of the empirical spectral distribution $\mu_{\mathtt{u}}^{(n)}$, is the semicircle distribution:

$$\mathrm{SC}(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\, \mathbb{1}_{[-2,2]}(x)\, dx. \tag{2.2}$$

The Laguerre ensemble. For all $n \geq 1$, let $N = N(n)$ be such that $n \leq N$. Let X be a $n \times N$ complex matrix with independent entries having a gaussian distribution such that $\mathbb{E}X_{ij} = \mathbb{E}(X_{ij}^2) = 0$ and $\mathbb{E}|X_{ij}|^2 = 1$. Then, the Laguerre Unitary Ensemble $\mathrm{LUE}_N(n)$ is the distribution of $XX^\star$, which is proportional to $(\det XX^\star)^{N-n} \exp(-\mathrm{Tr}\, XX^\star)$. The law of the rescaled matrix $L = \frac{1}{N} XX^\star$ is therefore given by

$$\mathbb{Q}_1^{(n)}(dL) := \frac{N^{-n^2}}{\tilde{\Gamma}_n(N)} (\det L)^{N-n} \exp\left(-N \mathrm{Tr}\, L\right) dL\,, \tag{2.3}$$

where $\tilde{\Gamma}_n(N)$ is the multigamma function. During this article, we will assume that $N/n \to \tau^{-1} > 1$ as $n \to +\infty$. The equilibrium measure of this Laguerre ensemble, i.e. the limit of the empirical spectral measure $\mu_{\mathtt{u}}^{(n)}$, is the Marchenko-Pastur distribution with parameter τ:

$$\mathrm{MP}_\tau(dx) = \frac{\sqrt{(\tau^+ - x)(x - \tau^-)}}{2\pi\tau x}\ \mathbb{1}_{(\tau^-, \tau^+)}(x) dx \tag{2.4}$$

where $\tau^{\pm} := (1 \pm \sqrt{\tau})^2$.

The Gross-Witten ensemble. Our third model has its eigenvalues on $\mathbb{T}$ and corresponds to the Gross-Witten measure $\mathbb{GW}_{\mathtt{g}}^{(n)}$ with parameter $\mathtt{g} \in \mathbb{R}$. It is a probability measure on $\mathbb{U}(n)$ which is absolutely continuous with respect to the Haar measure $\mathbb{P}^{(n)}$ on the unitary group $\mathbb{U}(n)$, with density:

$$\frac{d\mathbb{GW}_{\mathtt{g}}^{(n)}}{d\mathbb{P}^{(n)}}(U) := \frac{1}{\mathcal{Z}_n(\mathtt{g})} \exp\left[\frac{n\mathtt{g}}{2} \mathrm{Tr}(U + U^\star)\right]. \tag{2.5}$$

Let us mention that the Gross-Witten measure arises in the context of the Ulam's problem which concerns the length of the longest increasing subsequence inside a uniform permutation [3]. For other details and applications of this distribution we refer to [20] p. 203, [19], [33].

There are two different behaviors according to the value of the parameter $\mathtt{g}$.

For $|\mathtt{g}| \leq 1$ (ungapped or strongly coupled phase). In this context, the equilibrium measure $\mathrm{GW}_{\mathtt{g}}$ is supported by $\mathbb{T}$ and has the following density:

$$\mathrm{GW}_{\mathtt{g}}(dz) = \frac{1}{2\pi}(1 + \mathtt{g}\cos\theta)\, d\theta,\ (z = e^{\mathrm{i}\theta}, \theta \in [-\pi, \pi)). \tag{2.6}$$

Note that $\mathrm{GW}_{\mathtt{g}}$ has only nontrivial moments of order ± 1.

For $|\mathtt{g}| > 1$, the equilibrium measure is supported by an arc. This case will not be considered here since the paper would be lenghtened with involved computations.

3. Recap on Orthogonal polynomials

In this section we recall the possible parametrization of positive measures on $\mathbb{R}$ (resp. $\mathbb{T}$) by their Jacobi (resp. Verblunsky) coefficients. The latter appear through the spectral theory of orthogonal polynomials on the real line (OPRL), resp. the spectral theory of orthogonal polynomials on the unit circle (OPUC), which we briefly recall here. In the next section, we will use these parametrizations in order to recall the large deviations principles satisfied by the spectral measures of the models defined in Section 2.

3.1. **OPRL.** Let ρ be a positive measure on $\mathbb{R}$ whose support is bounded but not made of a finite union of points. Let $(p_n(x))_{n\geq 0}$ be the sequence of orthonormal polynomials associated to ρ, obtained by applying the Gram-Schmidt algorithm to the basis $\{1, x, x^2, \ldots\}$. Then, there exists two sequences of uniformly bounded real numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that $a_n > 0$ for all $n \geq 0$ and such that the polynomials $p_n(x)$'s satisfy the following three terms recursion:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x). \tag{3.1}$$

The parameters $\{a_n, b_n\}_{n=1}^{\infty}$ are called the *Jacobi parameters* associated to ρ. We will denote

$$\mathrm{Jac}(\rho) = \begin{pmatrix} b_1, & b_2, & \cdots \\ a_1, & a_2, & \cdots \end{pmatrix}. \tag{3.2}$$

As it is well known (see, e.g., [29, Section 1.3]), Equation (3.1) sets up the one-to-one correspondence between uniformly bounded sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and positive measures ρ on $\mathbb{R}$ whose supports are bounded but not made of a finite union of points. Moreover, a similar argument implies that there exists a one-to-one correspondence between the set of positive measures ρ on $\mathbb{R}$ whose support are finite union of N distinct points and the set of sequences $(a_n)_{1\leq n\leq N-1}$ and $(b_n)_{1\leq n\leq N}$ such that $a_n > 0$ for all $1 \leq n \leq N$. Let us mention that the Jacobi parameters of the semicircle law are given by:

$$\mathrm{Jac}(\mathrm{SC}) = \begin{pmatrix} 0, & 0, & \cdots \\ 1, & 1, & \cdots \end{pmatrix}, \tag{3.3}$$

it is called the "free" case in the OPRL literature.

When ρ is supported on $[0, \infty)$ the recursion coefficients can be decomposed as

$$\begin{aligned} b_k &= z_{2k-2} + z_{2k-1}, \\ a_k^2 &= z_{2k-1} z_{2k}, \end{aligned} \tag{3.4}$$

for $k \geq 1$, where $z_k \geq 0$ and $z_0 = 0$. In fact, by Favard's Theorem a measure ρ is supported on $[0, \infty)$ if and only if its Jacobi coefficients satisfy the decomposition (3.4). In particular, the MP_τ distribution corresponds to $z_{2n-1}^{\mathrm{MP}} = 1$ and $z_{2n}^{\mathrm{MP}} = \tau$ for all $n \geq 1$, so that

$$\mathrm{Jac}(\mathrm{MP}_\tau) = \begin{pmatrix} 1, & 1+\tau, & 1+\tau, & \cdots \\ \sqrt{\tau}, & \sqrt{\tau}, & \sqrt{\tau}, & \cdots \end{pmatrix}. \tag{3.5}$$

Let us finally mention that the measure ρ can be realized as the spectral measure associated to the pair (J, e_1), where J is the so-called Jacobi matrix which represents the multiplication by x in the basis $(p_n(x))_{n\geq 0}$:

$$J := \begin{pmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & \ddots & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.6}$$

3.2. **OPUC.** Let μ be a probability measure on $\mathbb{T}$ whose support is not a finite set of points. Let $(\varphi_n(z))_{n\geq 0}$ be the sequence of orthonormal polynomials associated to μ, obtained by applying the Gram-Schmidt algorithm to the basis $\{1, z, z^2, \ldots\}$. Then, there exists a sequence of complex numbers $(\alpha_n)_{n\geq 0}$, called the *Verblunsky coefficients* associated to μ, such that $|\alpha_n| < 1$ for all $n \geq 0$ and such that the polynomials $\varphi_n(z)$'s satisfy the following recursion:

$$z\varphi_n(z) = \rho_n \varphi_{n+1}(z) + \bar{\alpha}_n \varphi_n^*(z), \tag{3.7}$$

where

$$\varphi_n^*(z) = z^n\, \overline{\varphi_n(1/\bar{z})} \qquad \rho_n = (1 - |\alpha_n|^2)^{1/2}\,. \tag{3.8}$$

Equation (3.7) sets up a one-to-one correspondence between sequences $(\alpha_n)_{n\geq 0}$ with values inside $\{z, |z| < 1\}$ and the set of positive measures μ on $\mathbb{T}$ whose supports are not finite union of points. Moreover, a similar argument implies that there exists a one-to-one correspondence between the set of positive measures μ on $\mathbb{T}$ whose support are finite union of N distinct points and the set of sequences $(\alpha_n)_{1\leq n\leq N}$ such that $|\alpha_n| < 1$ for all $0 \leq n \leq N-1$ and $|\alpha_N| = 1$.

The sequence $\alpha_n \equiv 0$ corresponds to λ_0, the normalized Lebesgue measure on $\mathbb{T}$, and is called "free" case in the OPUC literature.

For the Gross-Witten model, when $|\mathsf{g}| \leq 1$, the V-coefficients are given by (see [29], p. 86):

$$\alpha_n^{\mathrm{GW}} = \begin{cases} -\dfrac{x_+ - x_-}{x_+^{n+2} - x_-^{n+2}} & \text{if } |\mathsf{g}| < 1 \\ \dfrac{(-\mathsf{g})^{n+1}}{n+2} & \text{if } |\mathsf{g}| = 1\,, \end{cases} \tag{3.9}$$

where $x_\pm$ are roots of the equation

$$x + \frac{1}{x} = -\frac{2}{\mathsf{g}}\,.$$

In particular

$$\alpha_0^{\mathrm{GW}} = \frac{\mathsf{g}}{2}\,. \tag{3.10}$$

4. Recap on LDP and sum rules

See [11, 12, 1] for background on LDP. For the self-adjoint models the sequence $(\mu_{\mathtt{u}}^{(n)})$ satisfies the LDP at scale n^2 with good rate function involving the logarithmic entropy and the potential. Moreover the extremal eigenvalues satisfy the LDP at scale n with a rate function $\mathcal{F}_H$ and $\mathcal{F}_L^{\pm}$, which represent effective potentials. For the unitary model studied here, the support of the limiting measure is the whole unit circle and there exists no outlier. The following results are the sum rules obtained by the second author of this paper together with Gamboa and Nagel.

4.1. **OPRL.**

4.1.1. *LDP on the measure side.* To begin with, let us give some notations. Let $\mathcal{S} = \mathcal{S}(\alpha^-, \alpha^+)$ be the set of all bounded positive measures μ on $\mathbb{R}$ with

(i) $\mathrm{supp}(\mu) = J \cup \{E_i^-\}_{i=1}^{N^-} \cup \{E_i^+\}_{i=1}^{N^+}$, where $J \subset I = [\alpha^-, \alpha^+]$, $N^-, N^+ \in \mathbb{N} \cup \{\infty\}$ and
$$E_1^- < E_2^- < \cdots < \alpha^- \quad \text{and} \quad E_1^+ > E_2^+ > \cdots > \alpha^+.$$

(ii) If N^- (resp. N^+) is infinite, then E_j^- converges towards α^- (resp. E_j^+ converges to α^+).

Such a measure μ will be written as
$$\mu = \mu_{|I} + \sum_{i=1}^{N^+} \gamma_i^+ \delta_{E_i^+} + \sum_{i=1}^{N^-} \gamma_i^- \delta_{E_i^-}, . \tag{4.1}$$

Further, we define $\mathcal{S}_1 = \mathcal{S}_1(\alpha^-, \alpha^+) := \{\mu \in \mathcal{S} |\, \mu(\mathbb{R}) = 1\}$. We endow $\mathcal{S}_1$ with the weak topology and the corresponding Borel σ-algebra.

On the measure side we have

Theorem 4.1. *Under* GUE(n) *(resp.* LUE$_N(n)$*) the family of distributions of* $(\mu_{\mathtt{w}}^{(n)})$ *satisfies the LDP on* $\mathcal{M}_1(\mathbb{R})$ *in the scale* n *with good rate function* $\mathcal{I}_{\mathrm{meas}}^H$ *(resp.* $\mathcal{I}_{\mathrm{meas}}^L$*) given by*
$$\mathcal{I}_{\mathrm{meas}}^H(\mu) = \begin{cases} \mathcal{K}(\mathrm{SC} \mid \mu) + \sum_k \mathcal{F}_H(E_k^\pm) & \text{if } \mu \in \mathcal{S}_1(-2,2), \\ \infty & \text{otherwise}, \end{cases} \tag{4.2}$$
where
$$\mathcal{F}_H(x) := \begin{cases} \displaystyle\int_2^{|x|} \sqrt{t^2-4}\, dt & \text{if } |x| \geq 2 \\ \infty & \text{otherwise}, \end{cases} \tag{4.3}$$
resp.
$$\mathcal{I}_{\mathrm{meas}}^L(\mu) = \begin{cases} \mathcal{K}(\mathrm{MP}_\tau \mid \mu) + \sum_k \mathcal{F}_L^\pm(E_k^\pm) & \text{if } \mu \in \mathcal{S}_1(\tau^-, \tau^+) \\ \infty & \text{otherwise}, \end{cases} \tag{4.4}$$
where
$$\mathcal{F}_L^+(x) = \int_{\tau^+}^x \frac{\sqrt{(t-\tau^-)(t-\tau^+)}}{t\tau}\, dt \quad \text{if } x \geq \tau^+, \tag{4.5}$$
$$\mathcal{F}_L^-(x) = \int_x^{\tau^-} \frac{\sqrt{(\tau^- - t)(\tau^+ - t)}}{t\tau}\, dt \quad \text{if } x \leq \tau^-, \tag{4.6}$$
$$\mathcal{F}_L^\pm(x) = \infty \quad \text{if } x \in [\tau^-, \tau^+]\,. \tag{4.7}$$
The measure SC *(resp.* MP$_\tau$*) is the unique minimum of* $\mathcal{I}_{\mathrm{meas}}^H$ *(resp.* $\mathcal{I}_{\mathrm{meas}}^L$*).*

4.1.2. *LDP on the coefficients side.* We start by stating the classical Killip-Simon sum rule (due to [22] and explained in [32] p.37). It gives two different expressions for the discrepancy between a measure and to the semicircle law SC.

For a probability measure μ on $\mathbb{R}$ with recursion coefficients $\mathfrak{a} := (a_k)_k$, $\mathfrak{b} := (b_k)_k$, define
$$\mathcal{I}_{\mathrm{coeff}}^H(\mathfrak{a}, \mathfrak{b}) := \sum_{k \geq 1} \left(\frac{1}{2} b_k^2 + G(a_k^2) \right) , \tag{4.8}$$

where $G(x) = x - 1 - \log x$. It is a convex function of $(\mathfrak{a}, \mathfrak{b})$ which has a unique minimum at $a_k \equiv 1, b_k \equiv 0$, corresponding to the semicircle law SC (see (3.3)).

If the support of μ is a subset of $[0, \infty)$ with $\mathfrak{z} = (z_k)_k$, define

$$\mathcal{I}^L_{\text{coeff}}(\mathfrak{z}) := \sum_{k=1}^{\infty} \tau^{-1} G(z_{2k-1}) + G(\tau^{-1} z_{2k}) \,. \tag{4.9}$$

It is a convex function of $\mathfrak{z}$ which has a unique minimum at $\mathfrak{z} = (z_k)_{k \geq 0}$ with

$$z_0 = 0, Z_{2k-1} = 1, z_{2k} = \tau \ (k \geq 1) \,.$$

which corresponds with MP_τ.

Then we have the following theorem.

Theorem 4.2 ([22]). (1) *Let J be a Jacobi matrix with diagonal entries $b_k \in \mathbb{R}$ and subdiagonal entries $a_k > 0$ satisfying $\sup_k a_k + \sup_k |b_k| < \infty$ and let μ be the associated spectral measure. Then $\mathcal{I}^H_{meas}(\mu) = \infty$ if $\mu \notin \mathcal{S}_1(-2, 2)$ and for $\mu \in \mathcal{S}_1(-2, 2)$,*

$$\mathcal{I}^H_{\text{coeff}}(\mathfrak{a}, \mathfrak{b}) = \mathcal{I}^H_{\text{meas}}(\mu) \tag{4.10}$$

where in (4.10), both sides may be infinite simultaneously.

(2) *Assume the entries of the Jacobi matrix J satisfy the decomposition (3.4) with $\sup_k z_k < \infty$ and let μ be the spectral measure of J. Then for all $\tau \in (0, 1]$, $\mathcal{I}_L(\mu) = \infty$ if $\mu \notin \mathcal{S}_1(\tau^-, \tau^+)$ and for $\mu \in \mathcal{S}_1(\tau^-, \tau^+)$,*

$$\mathcal{I}^L_{\text{coeff}}(\mathfrak{z}) = \mathcal{I}^L_{\text{meas}}(\mu) \tag{4.11}$$

where in (4.11), both sides may be infinite simultaneously.

Note that if $\tau = 1$, the support of the limit measure is $[0, 4]$, so that we have a hard edge at 0 with $N^- = 0$ and no contribution of outliers to the left.

4.2. **OPUC.** For the unitary case we have LDPs on the measure side and some sum rules. In the following $\mathcal{K}(\nu \mid \mu)$ denotes the Kullback-Leibler divergence or relative entropy of ν with respect to μ on $\mathbb{T}$.

4.2.1. *Measure side.*

Theorem 4.3 ([14] Cor. 4.5). *Under $\mathbb{GW}^{(n)}_{\mathsf{g}}$, with $|\mathsf{g}| \leq 1$, the sequence of spectral measures $\mu^{(n)}$ satisfies the LDP in $\mathcal{M}_1(\mathbb{T})$ with speed n and rate function*

$$\mathcal{I}^{\text{GW}}_{meas}(\mu) = \mathcal{K}(\text{GW}_{\mathsf{g}} \mid \mu) \,.$$

The measure GW_{g} is the unique minimum of $\mathcal{I}^{GW}$.

4.2.2. *Coefficient side - sum rules.* For a probability measure μ on $\mathbb{T}$ we denote by $\boldsymbol{\alpha} := (\alpha_k)_k$ the sequence of its Verblunsky coefficients.

On the unit circle, the most famous sum rule is the Szegő formula:

$$\mathcal{K}(\lambda_0 \mid \mu) = -\sum_{k \geq 0} \log(1 - |\alpha_k|^2) \tag{4.12}$$

where, as above λ_0 is the normalized Lebesgue measure on $\mathbb{T}$, whose Verblunsky coefficients are $\alpha_k = 0$ for every k.

There are many proofs of (4.12) in [29] and a probabilistic proof in [17].

In the Gross-Witten case, we define, for $|\mathtt{g}| \leq 1$

$$\mathcal{I}_{\rm coeff}^{\rm GW}(\alpha) := \mathcal{K}(\mathrm{GW}_{\mathtt{g}} \mid \lambda_0) - \mathtt{g}\Re\left(\alpha_0 - \sum_{k=1}^{\infty} \alpha_k \bar{\alpha}_{k-1}\right) + \sum_{k=0}^{\infty} - \log(1 - |\alpha_k|^2)\,. \tag{4.13}$$

where

$$\mathcal{K}(\mathrm{GW}_{\mathtt{g}} \mid \lambda_0) := 1 - \sqrt{1-\mathtt{g}^2} + \log\frac{1+\sqrt{1-\mathtt{g}^2}}{2}\,. \tag{4.14}$$

The following sum rule was pointed out in [29] Theorem 2.8.1 for GW_{-1} and extended in Cor. 5.4 in [14]) for $-1 < \mathtt{g} < 0$, but the proof remains valid for $0 < \mathtt{g} \leq 1$. In [8], the authors proved the LDP for the coefficient side when $\mathtt{g} = -1$ by probabilistic arguments, and actually this proof may be extended easily to the case $|\mathtt{g}| < 1$.

Theorem 4.4. *Let μ be a probability measure on $\mathbb{T}$ with Verblunsky coefficients $\boldsymbol{\alpha} = (\alpha_k)_{k\geq 0} \in \mathbb{D}^{\mathbb{N}}$. Then, for $0 \leq |\mathtt{g}| < 1$, we have*

$$\mathcal{I}_{\rm coeff}^{\rm GW}(\boldsymbol{\alpha}) = \mathcal{I}_{\rm meas}^{\rm GW}(\mu)\,. \tag{4.15}$$

Remark 1. The case $|\mathtt{g}| > 1$ is more complex (see Conjecture in [14]).

5. LDP for perturbations

We are now in position to state and prove our main results, which are concerned with large deviations of spectral measures of rank one perturbation of the models introduced in Section 2. As advertised during the Introduction, we will always provide two proofs. The first proof, which will be called the *direct proof*, uses the fact that the law of the spectral measure of the deformed model is a tilted version of the law of the initial model. The second proof, which will be called the *alternative proof*, uses the fact that the Jacobi (resp. Verblunsky) coefficients of the deformed models are simple perturbations of the initial coefficients (in fact, only one parameter is affected).

5.1. **Additive perturbation - Gaussian case.** For all $n \geq 1$, let us consider

$$W_n = \frac{1}{\sqrt{n}} X_n + A_n,$$

where X_n follows the GUE(n) distribution and A_n is a rank-one deterministic matrix of size $n \times n$. Since the Gaussian Unitary Ensemble is unitarily invariant, we can assume that $A_n = \theta u u^*$, where $\theta \in \mathbb{R}$ and where $u = e_1$ is the first vector of the canonical basis. Let $\mu_{\mathtt{W}}^{(n)}$ be the spectral measure of the pair (W_n, u). It is known ([24] Th. 4.6, [26] Cor. 1) that, as $n \to \infty$, $\mu_{\mathtt{W}}^{(n)}$ converges in probability towards the following probability measure:

$$\mu_{\mathrm{SC},\theta}(dx) = \frac{\sqrt{(4-x^2)_+}}{2\pi(\theta^2+1-\theta x)}\, dx + \left(1 - \frac{1}{\theta^2}\right)_+ \delta_{\theta+\frac{1}{\theta}}\,. \tag{5.1}$$

Our first result establishes a large deviation principle for the sequence of probability measures $(\mu_{\mathtt{W}}^{(n)})_{n\geq 1}$.

Theorem 5.1. *The family* $(\mu_{\mathtt{w}}^{(n)})$ *satisfies the LDP at scale* n *with good rate function*

$$\mathcal{I}^W(\mu) = \begin{cases} \mathcal{K}(\mathrm{SC} \mid \mu) - \theta m_1(\mu) + \frac{1}{2}\theta^2 + \sum_k \mathcal{F}(E_k^{\pm}) & \text{if } \mu \in \mathcal{S}_1(-2,2), \\ \infty & \text{otherwise.} \end{cases} \tag{5.2}$$

Moreover:

(1) $\mu_{\mathrm{SC},\theta}$ *is the unique minimizer of* $\mu \mapsto \mathcal{I}^W(\mu)$,
(2) $\mu_{\mathrm{SC},\theta}$ *is the unique minimizer of* $\mu \mapsto \mathcal{I}^H_{\mathrm{meas}}(\mu)$ *under the constraint* $m_1(\mu) = \theta$, *where we recall that* $\mathcal{I}^H_{\mathrm{meas}}$ *is defined in* (4.2).

Proof. We first prove (5.2) using two different arguments.

A) Direct proof. If X has the GUE(n) distribution (see (2.1)), the distribution of $W = n^{-1/2}X + \theta uu^*$ is

$$\mathbb{P}_\theta^{(n)}(dW) = \frac{1}{\mathcal{Z}_0^{(n)}} \exp\left(-\frac{n}{2}\mathrm{Tr}((W-\theta uu^*)(W^\star - \theta uu^*))\right) dW.$$

But

$$\begin{aligned}\mathrm{Tr}((W-\theta uu^*)&(W^\star - \theta uu^*) \\ &= \mathrm{Tr}(WW^\star) - \theta\left(\mathrm{Tr}(Wuu^*) + \mathrm{Tr}(uu^*W^\star)\right) + \theta^2\mathrm{Tr}(uu^*) \\ &= \mathrm{Tr}(WW^\star) - 2\theta u^*Wu + \theta^2,\end{aligned}$$

which allows us to rewrite

$$\mathbb{P}_\theta^{(n)}(dW) = \exp\left(-\frac{n}{2}(\theta^2 - 2\theta u^*Wu)\right)\ \mathbb{P}_0^{(n)}(dW)\,.$$

Let us finally notice that, since $u = e_1$, one has $u^*Wu = W_{11} = m_1(\mu_{\mathtt{w}}^{(n)})$, which yields

$$\mathbb{P}_\theta^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu) = \frac{\exp n\Psi(\mu)}{\mathbb{E}_0 \exp n\Psi(\mu_{\mathtt{w}}^{(n)})}\mathbb{P}_0^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu),$$

where

$$\Psi(\mu) = \theta m_1(\mu).$$

It remains to apply Varadhan's lemma (see [11] Th 4.3.1 or [12] Exercise 2.1.24). Notice that the uniform exponential integrability condition is satisfied since under $\mathbb{P}_0^{(n)}$, $m_1(\mu_{\mathtt{w}}^{(n)}) = X_{11} \sim \mathcal{N}(0; n^{-1})$, which implies that for every $\gamma > 0$,

$$\frac{1}{n}\log \mathbb{E}[\exp \gamma n m_1(\mu_{\mathtt{w}}^{(n)})] = \frac{\gamma^2}{2}\,.$$

B) Alternative proof. Fix $n \geq 1$. A consequence of the tridiagonal representation of the GUE(n) of Dumitriu and Edelman is that $\mu_{\mathtt{w}}^{(n)}$ is the spectral measure of the pair (J_n, e_1), where J_n is the following random Jacobi matrix:

$$J_n \sim \begin{pmatrix} \mathcal{N}(0,\frac{1}{n})+\theta & \frac{1}{\sqrt{n}}\chi_{2(n-1)} & & & \\ \frac{1}{\sqrt{n}}\chi_{2(n-1)} & \mathcal{N}(0,\frac{1}{n}) & \frac{1}{\sqrt{n}}\chi_{2(n-2)} & & \\ & \frac{1}{\sqrt{n}}\chi_{2(n-2)} & \mathcal{N}(0,\frac{1}{n}) & \ddots & \\ & & \ddots & \ddots & \frac{1}{\sqrt{n}}\chi_2 \\ & & & \frac{1}{\sqrt{n}}\chi_2 & \mathcal{N}(0,\frac{1}{n}) \end{pmatrix}.$$

Here, the matrix J_n is symmetric and up to this symmetry, its coefficients are independent. Note that this corresponds to the usual tridiagonalisation of the GUE(n) except for the addition of the parameter θ to the $(1,1)$ coefficient.

Fix μ a measure on $\mathbb{R}$ with Jacobi parameters $\mathfrak{a} = (a_n)_{n\geq 1}$ and $\mathfrak{b} = (b_n)_{n\geq 1}$. Then, using a projective method and the independence of the coefficients of J_n, we see that the rate function for the LDP on the coefficient side is given by

$$\mathcal{I}^W_{\text{coeff}}(\mathfrak{a},\mathfrak{b}) = \frac{1}{2}(b_1-\theta)^2 + \frac{1}{2}\sum_{k\geq 2} b_k^2 + + \sum_{k\geq 1} G(a_k^2) \tag{5.3}$$

$$= \frac{1}{2}\theta^2 - b_1\theta + \mathcal{I}^H_{\text{coeff}}(\mathfrak{a},\mathfrak{b})\,. \tag{5.4}$$

But, by Theorem 4.2,

$$\mathcal{I}^H_{\text{coeff}}(\mathfrak{a},\mathfrak{b}) = \mathcal{I}^H_{\text{meas}}(\mu) = \mathcal{K}(\text{SC} \mid \mu) + \sum_k \mathcal{F}(E_k^\pm)\,.$$

Besides, $b_1 = m_1(\mu)$, so that the random measure $\mu_{\mathtt{w}}^{(n)}$ satisfies the LDP at scale n with rate function

$$\mathcal{I}^W(\mu) = \mathcal{I}^W_{\text{coeff}}(\mathfrak{a},\mathfrak{b}) = \mathcal{K}(\text{SC} \mid \mu) - \theta m_1(\mu) + \frac{1}{2}\theta^2 + \sum_k \mathcal{F}(E_k^\pm)\,. \tag{5.5}$$

We now turn to the proofs of (1) and (2).

(1) The infimum can be looked from the coefficient side, i.e. from (5.5) and (5.3), and is given by:

$$\text{Jac}(\operatorname{argmin} \mathcal{I}^W) = \begin{pmatrix} \theta, & 0, & 0, & \cdots \\ 1, & 1, & 1, & \cdots \end{pmatrix}.$$

Using section 7.2, we deduce that $\text{Jac}(\operatorname{argmin}\mathcal{I}^W) = \text{Jac}(\tilde{\mu}_{-\theta,0})$, namely $\operatorname{argmin}\mathcal{I}^W = \mu_{\text{SC},\theta}$.

(2) If $m_1(\mu) = \theta$, we deduce from (5.4) and the sum rules that

$$\mathcal{I}^H_{\text{meas}}(\mu) = \mathcal{I}^W_{\text{meas}}(\mu) + \frac{\theta^2}{2} \geq \frac{\theta^2}{2}\,,$$

with equality if and only if $\mu = \mu_{\text{SC},\theta}$.

□

Remarks. (1) Let us first observe that we can check by hands that the minimum of $\mathcal{I}^W$ is zero. Indeed, since $\mathcal{F}_H$ is also given by[1]

$$\mathcal{F}_H(\theta+\theta^{-1}) = \frac{(\theta+\theta^{-1})^2}{2} - 2\int \log(\theta+\theta^{-1}-x)\,\text{SC}(dx) - 1,$$

we deduce that

$$\begin{aligned}\mathcal{K}(\text{SC} \mid \mu_{\text{SC},\theta}) &= \int \log(1+\theta^2-\theta x)\,\text{SC}(dx)\\ &= \log\theta + \int \log(\theta+\theta^{-1}-x)\,\text{SC}(dx)\\ &= \log\theta + \frac{(\theta+\theta^{-1})^2}{4} - \frac{1}{2} - \frac{1}{2}\mathcal{F}(\theta+\theta^{-1})\end{aligned}$$

[1] There is a mistake in [1] p.81 see http://www.wisdom.weizmann.ac.il/ zeitouni/cormat.pdf

$$= \log\theta + \frac{(\theta+\theta^{-1})^2}{4} - \frac{1}{2} - \frac{1}{4}(\theta^2-\theta^{-2}) + \log\theta$$
$$= \frac{(\theta^{-2}-\theta^2)}{2} + 2\log\theta + \frac{\theta^2}{2}.$$

Hence $\mathcal{K}(\mathrm{SC} \mid \mu_{\mathrm{SC},\theta}) + \mathcal{F}(\theta+\theta^{-1}) = \frac{\theta^2}{2}$ and:

$$\mathcal{I}^W_{\mathrm{meas}}(\mu_{\mathrm{SC},\theta}) = \mathcal{K}(\mathrm{SC} \mid \mu_{\mathrm{SC},\theta}) + \mathcal{F}(\theta+\theta^{-1}) - \theta m_1(\mu_{\mathrm{SC},\theta}) + \frac{\theta^2}{2}$$
$$= \frac{\theta^2}{2} - \theta^2 + \frac{\theta^2}{2} = 0.$$

(2) The fact that $\mu_{SC,\theta}$ is the only minimizer of $\mathcal{I}^W$ allows to retrieve the convergence of $\mu_{\mathtt{w}}^{(n)}$ towards $\mu_{\theta,\mathrm{SC}}$, and actually to strengthen the convergence in probability into an almost sure convergence.

5.2. **Multiplicative perturbation.** For all $n \geq 1$, let us consider

$$S_n = \frac{1}{n}\Sigma_n^{1/2} L_n \Sigma_n^{1/2}$$

where $\Sigma_n = \mathrm{Diag}(\theta, 1, \dots)$. It is known ([23] Prop. 6.1) that the sequence of measure $(\mu_{\mathtt{w}}^{(n)})_{n\geq 1}$ has a limit. An explicit computation of the limiting measure $\mu^{L,\theta}$ can be performed as in [26], and we get[2]

$$\mu^L(dx) = \frac{\sqrt{4\tau - (x-(1+\tau))^2}}{2\pi x\left((\theta+\tau-1) + x(\theta^{-1}-1)\right)}dx + u\delta_0 + v\delta_w, \tag{5.6}$$

with

$$u = \frac{(\tau-1)_+}{\theta+\tau-1},\ v = \frac{\tau\left((\theta-1)^2-\tau\right)_+}{(\theta-1)(\theta+\tau-1)},\ w = -\frac{\theta+\tau-1}{\theta^{-1}-1}.$$

Here, we will restrict the setting to the case where $n/N \to \tau < 1$, that is to the case where μ^L does not have a mass at zero. In this context, we obtain a large deviation principle for the family of spectral measures $(\mu_{\mathtt{w}}^{(n)})_{n\geq 1}$ associated to the pairs (S_n, e_1).

Theorem 5.2. *The family $(\mu_{\mathtt{w}}^{(n)})$ satisfies the LDP at scale n with good rate function*

$$\mathcal{I}^S(\mu) = \begin{cases} \mathcal{K}(\mathrm{MP}_\tau \mid \mu) + \frac{\theta^{-1}-1}{\tau} m_1(\mu) + \frac{1}{\tau}\log\theta + \sum_k \mathcal{F}_L(E_k^\pm) & \text{if } \mu \in \mathcal{S}_1(\tau^-,\tau^+) \\ \infty & \text{otherwise.} \end{cases} \tag{5.7}$$

Moreover,

(1) *$\mu^{L,\theta}$ is the unique minimizer of $\mu \mapsto \mathcal{I}^S(\mu)$,*
(2) *$\mu^{L,\theta}$ is the unique minimizer of $\mathcal{I}^L(\mu)$ under the constraint $m_1(\mu) = \theta$.*

Proof. We first provide two proofs of (5.7).

A) Direct proof. Let L be a random $n \times n$ matrix following the $\mathrm{LUE}_N(n)$ distribution (see (2.3)), and Σ a Hermitian positive $n \times n$. Then, the distribution

[2]This is the same measure as $\mu_{\mathrm{MP},\tau^{-1},\theta}$ in [26] up to a little change, due to the convention on the definition of sample covariance matrix.

of $S = \Sigma^{1/2} L \Sigma^{1/2}$ is given by the following density:

$$\frac{1}{\tilde{\Gamma}_n(N)} N^{-n^2} (\det S)^{N-n} (\det \Sigma)^{-N} \exp -N \mathrm{Tr}\, \Sigma^{-1} S \; dS.$$

In our case, $N = \tau^{-1} n$ and $\Sigma_n = \mathrm{Diag}(\theta, 1, \dots, 1)$, so that we have $\det \Sigma_n = \theta$ and:

$$\mathrm{Tr}(S_n \Sigma_n^{-1}) = \theta^{-1} (S_n)_{11} + \sum_{k \geq 2} (S_n)_{kk} = (\theta^{-1} - 1)(S_n)_{11} + \mathrm{Tr}\, S_n.$$

Therefore, the law of $S_n = \Sigma_n^{1/2} L_n \Sigma_n^{1/2}$ is

$$\begin{aligned} &\mathbb{Q}_\theta^{(n)}(dS) \\ &\quad = \frac{(n\tau^{-1})^{n^2 \tau^{-1}}}{\tilde{\Gamma}_n(n\tau^{-1})} \theta^{-n\tau^{-1}} (\det S)^{n(\tau^{-1}-1)} \exp\left(-n\tau^{-1}(\mathrm{Tr} S + (\theta^{-1} - 1) S_{11})\right) \; dS \end{aligned}$$

Moreover, since $\mu_{\mathtt{w}}^{(n)}$ is the spectral measure associated to the pair (S_n, e_1), we have $S_{11} = m_1(\mu_{\mathtt{w}}^{(n)})$, which implies that

$$\mathbb{Q}_\theta^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu) = \frac{\exp n\Phi(\mu)}{\mathbb{Q}_1^{(n)}(\exp n\Phi(\mu_{\mathtt{w}}^{(n)})} \mathbb{Q}_1^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu),$$

where

$$\Phi(\mu) = \tau^{-1}(1 - \theta^{-1}) m_1(\mu).$$

In order to apply Varadhan's Lemma, let us check the uniform exponential integrability condition. Under $\mathbb{Q}_1^{(n)}$,

$$n\tau^{-1} m_1(\mu_{\mathtt{w}}^{(n)}) = n\tau^{-1} S_{11} \overset{(d)}{=} \chi^2(n\tau^{-1}),$$

which implies that for all $\varphi < 1$,

$$\mathbb{Q}_1^{(n)}(\exp \varphi n \tau^{-1} S_{11}) = (1 - \varphi)^{-n\tau^{-1}}.$$

Therefore $\gamma \in (1, (1 - \theta^{-1})_+^{-1})$,

$$\frac{1}{n} \log \mathbb{Q}_1^{(n)} \exp \gamma n \Phi(\mu_{\mathtt{w}}^{(n)}) = -\tau^{-1} \log(1 - \gamma(1 - \theta^{-1})).$$

Thus, the uniform integrability condition is satisfied and we can apply Varadhan's Lemma, which leads to (5.7).

B) Alternative proof.

Fix $n \geq 1$. A consequence of the tridiagonal representation of the LUE(n) of Dumitriu and Edelman is that $\mu_{\mathtt{w}}^{(n)}$ is the spectral measure of the pair (J_n, e_1), where $J_n = B_n B_n^\star$ with:

$$B_n \sim \frac{1}{\sqrt{2N}} \begin{pmatrix} \chi_{2N} & & & & & \\ \sqrt{\theta} \cdot \chi_{2(n-1)} & \chi_{2(N-1)} & & & & \\ & \chi_{2(n-2)} & \chi_{2(N-2)} & & & \\ & & \ddots & \ddots & & \\ & & & \chi_2 & \chi_{2(N-n+1)} \end{pmatrix}.$$

Here, the matrix B_n is bidiagonal and its coefficients are independent. Note that this corresponds to the usual bidiagonal matrix of the LUE(n) except for the addition of the multiplicative factor $\sqrt{\theta}$ to the $(1,2)$ coefficient. Using the parameters system (3.4), we deduce that the transformation $L_n \mapsto S_n$ changes the first coefficient z_1 into $z_1' = \theta z_1$ and does not change the other parameters. Since the rate function for z_1 is $\tau^{-1}G(z)$ with $G(z) = z - 1 - \log z$, the rate function for θz_1 is $\tau^{-1}G(z/\theta)$. Let μ be a positive measure on $[0,\infty)$ with $\mathfrak{z}$-parameters $(z_i)_{i\geq 0}$. Then, using a projective method and the independence of the coefficients of J_n, we see that the LDP on the coefficient side is given by:

$$\begin{aligned}\mathcal{I}^S_{\text{coeff}}(\mathfrak{z}) &= \tau^{-1}\left[G(z_1/\theta) - G(z_1)\right] + \mathcal{I}^L_{\text{coeff}}(\mathfrak{z}) \\ &= \tau^{-1}(\theta^{-1}-1)z_1 + \tau^{-1}\log\theta + \mathcal{I}^L_{\text{coeff}}(\mathfrak{z})\,.\end{aligned}$$

But by the sum rule (4.11),

$$\mathcal{I}^L_{\text{coeff}}(\mathfrak{z}) = \mathcal{K}(\text{MP} \mid \mu) + \sum_k \mathcal{F}_L(E_k^\pm). \tag{5.8}$$

Moreover, $z_1 = b_1 = m_1(\mu)$, so that our random measure satisfies the LDP with rate function

$$\mathcal{I}^W_{\text{meas}}(\mu) = \mathcal{K}(\text{MP} \mid \mu) + \tau^{-1}(\theta^{-1}-1)m_1(\mu) + \tau^{-1}\log\theta + \sum_k \mathcal{F}_L(E_k^\pm). \tag{5.9}$$

We now turn to the proof of (1) and (2).

(1) The minimizer of $\mathcal{I}^S$ can be looked from the coefficient side and is given by the following $\mathfrak{z}$-parameters:

$$z_1 = \theta, z_{2k-1} = 1\ ,\ k\geq 2\ ,\ z_{2k} = \tau\ ,\ k\ \geq 1\,. \tag{5.10}$$

Owing to (3.4), it corresponds to the following Jacobi coefficients:

$$\text{Jac}(\operatorname{argmin} I^W) = \begin{pmatrix} \theta, & 1+\tau, & 1+\tau, & \cdots \\ \sqrt{\theta\tau}, & \sqrt{\tau}, & \sqrt{\tau}, & \cdots \end{pmatrix}. \tag{5.11}$$

By Lemma 7.1, we deduce that

$$\text{Jac}\left(T_{\sqrt{\theta\tau},\theta}(\operatorname{argmin} I^W)\right) = \text{Jac}(\mu_{b,c}) \tag{5.12}$$

where

$$b = \frac{1+\tau-\theta}{\sqrt{\theta\tau}}\ ,\ c = \frac{1-\theta}{\theta}\,. \tag{5.13}$$

Coming back to our distribution, we find the expression given in (5.6) for μ^L. For $\theta > 1$ (resp. $\theta < 1$), it is the free binomial (resp. free Pascal) distribution (see Section 7.2).

(2) The condition $m_1(\mu) = \theta$ rewrites $\mathfrak{z}_1 = \theta$. Combining (5.8) and the sum rule (4.11), we deduce that

$$\mathcal{I}^L(\mu) = \mathcal{I}^S(\mu) + \tau^{-1}(\theta - 1 - \log\theta) \geq \tau^{-1}(\theta - 1 - \log\theta)\,,$$

with equality if and only if $\mu = \mu^L$. □

5.3. **Perturbations of Unitary Matrices.** Up to our knowledge, there is only one type of perturbation of unitary matrices which was studied in relation with Verblunsky (in short "V") coefficients. If $U_n \in \mathbb{U}(n)$ and $e = e_1$ is cyclical, let as usual $(\alpha_k)_{k\geq 0}$ be the V-coefficients of the pair (U,e). Now, for any fixed element $e^{\mathrm{i}\varphi} \in \mathbb{T}$, we define

$$W_n = U_n Q_n\ ,\ \text{whith } Q_n = I_n + (e^{\mathrm{i}\varphi} - 1)e\langle e, \cdot\rangle\ . \tag{5.14}$$

Such a rank-one perturbation has been considered in Sections 1.3.9, 1.4.16, 3.2, and 4.5 of [29], 10.1, A.1.D and A.2.D of [30], see also [31].

If μ is the spectral measure of the pair (U_n, e) let us denote by $\tau_{e^{\mathrm{i}\varphi}}\mu$ the spectral measure of the pair (W_n, e). A usual tool for the study of a measure μ on $\mathbb{T}$ is its Caratheodory transform, which is the analog of the Stieltjes transform, defined by

$$F_\mu(z) = \int_0^{2\pi} \frac{e^{\mathrm{i}\theta} + z}{e^{\mathrm{i}\theta} - z} d\mu(\theta)\,. \tag{5.15}$$

Conversely, if $d\mu = w(\theta)d\lambda_0(\theta) + d\mu_s$, then

$$w(\theta) = \lim_{r\uparrow 1} \Re F_\mu(re^{\mathrm{i}\theta}) \tag{5.16}$$

and μ_s is supported by $\{\theta : \lim_{r\uparrow 1} \Re F_\mu(re^{\mathrm{i}\theta}) = \infty\}$ (see [29] (1.3.31)).

The mapping $(\mu \mapsto \tau_{e^{\mathrm{i}\varphi}}\mu)$ gives at level of Caratheodory transform :

$$F_{\tau_{e^{\mathrm{i}\varphi}}\mu} = \frac{(1 - e^{-\mathrm{i}\varphi}) + (1 + e^{-\mathrm{i}\varphi})F}{(1 + e^{-\mathrm{i}\varphi}) + (1 - e^{-\mathrm{i}\varphi})F}\,, \tag{5.17}$$

(see [29](1.3.90)), which implies, by the Schur recursion, the remarkable relation:

$$\alpha_k(\tau_{e^{\mathrm{i}\varphi}}\mu) = e^{-\mathrm{i}\varphi}\alpha_k(\mu)\ ,\ (k \geq 0)\,. \tag{5.18}$$

When φ is varying, it generates the so-called Aleksandrov family of measures.
In particular, if $\mathbb{P}_\varphi^{(n)}$ (resp. $\mathbb{P}_0^{(n)}$) denotes the distribution of W_n (resp. U_n), we have

$$\mathbb{P}_\varphi^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu) = \mathbb{P}_0^{(n)}\left(\mu_{\mathtt{w}}^{(n)} \in d(\tau_{e^{-\mathrm{i}\varphi}}\mu)\right)\,. \tag{5.19}$$

Here is our theorem which establishes a large deviation principle for the sequence of spectral measures associated to the pairs (W_n, e).

Theorem 5.3. *Assume* $|\mathsf{g}|\ \leq 1$.

(1) *The family of distribution of random measures* $(\mu_{\mathtt{w}}^{(n)})$ *under* $\mathbb{GW}_{\mathsf{g}}$ *satisfies the LDP on* $\mathcal{M}_1(\mathbb{T})$*, at scale* n *with good rate function*

$$\mathcal{I}^W(\mu) = \mathcal{I}^{GW}(\mu) - \mathsf{g}\Re\left((e^{-\mathrm{i}\varphi} - 1)m_1(\mu)\right)\,, \tag{5.20}$$

where $\mathcal{I}^{GW}$ *has been defined in Theorem 4.3.*

(2) *The unique minimizer of* $\mathcal{I}^W$ *is* $\mu^\varphi = \tau_{e^{-\mathrm{i}\varphi}}(\mathrm{GW}_{\mathsf{g}})$ *and*

$$d\mu^\varphi(\theta) = \frac{1}{2\pi} \frac{1 + \mathsf{g}\cos\theta}{1 - 2\mathsf{g}\sin\frac{\varphi}{2}\sin\left(\theta - \frac{\varphi}{2}\right) + \mathsf{g}^2\sin^2\frac{\varphi}{2}}\ d\theta\,. \tag{5.21}$$

(3) μ^φ *is the unique minimizer of* $\mathcal{I}_{\mathrm{meas}}^{\mathrm{GW}}$ *under the constraint* $m_1(\mu) = \frac{\mathsf{g}}{2}e^{\mathrm{i}\varphi}$.

Proof. (1) A) Direct proof .

The distribution of U_n is $\mathbb{GW}_{\mathsf{g}}^{(n)}$ i.e.

$$\mathbb{P}_0^{(n)}(dU) = \frac{1}{\mathcal{Z}_0^{(n)}} \exp\frac{n\mathsf{g}}{2}\mathrm{Tr}(U + U^\star)\ dU\,, \tag{5.22}$$

and then the distribution of W_n is

$$\mathbb{P}_\varphi^{(n)}(dW) = \frac{1}{\mathcal{Z}_0^{(n)}} \exp\frac{n\mathsf{g}}{2}\mathrm{Tr}(WQ_n^{-1} + (WQ_n^{-1})^\star)\ dW\,. \tag{5.23}$$

But

$$\mathrm{Tr}\,(WQ_n^{-1}) = \mathrm{Tr}\,W + (e^{-\mathrm{i}\varphi} - 1)W_{11}\ ,\ \mathrm{Tr}\,(WQ_n^{-1})^\star = \mathrm{Tr}\,W^\star + (e^{\mathrm{i}\varphi} - 1)\bar{W}_{11} \tag{5.24}$$

so that

$$\mathbb{P}_\varphi^{(n)}(dW) = \frac{1}{\mathcal{Z}_0^{(n)}} \exp \frac{n\mathtt{g}}{2} \left(\mathrm{Tr}\,(W + W^*) + \Re\,(e^{-\mathrm{i}\varphi} - 1)W_{11}\right) dW \tag{5.25}$$

and since $W_{11} = m_1(\mu_{\mathtt{w}}^{(n)})$ we get

$$\mathbb{P}_\varphi^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu)) = \exp n\mathtt{g}\Re\left(e^{-\mathrm{i}\varphi} - 1)m_1(\mu)\right) \mathbb{P}_0(\mu_{\mathtt{w}}^{(n)} \in d\mu))\,. \tag{5.26}$$

This yields (1) by application of Varadhan's lemma without any condition since $m_1(\mu_{\mathtt{w}}^{(n)}) \in \mathbb{D}$.

(1) B) An alternate proof

Under $\mathbb{GW}_{\mathtt{g}}^{(n)}$, the rate function for the LDP of the V-coefficients is $\mathcal{I}_{\text{coeff}}^{\text{GW}}$ given by (4.13). After a pushing forward by (5.19) the new rate function on the coefficient side becomes

$$\mathcal{I}_{\text{coeff}}^{W}(\boldsymbol{\alpha}) = \mathcal{I}_{\text{coeff}}^{\text{GW}}(e^{\mathrm{i}\varphi}\boldsymbol{\alpha}) = \mathcal{I}_{\text{coeff}}^{\text{GW}}(\boldsymbol{\alpha}) - \mathtt{g}\Re\left(\alpha_0(e^{\mathrm{i}\varphi} - 1)\right)\,. \tag{5.27}$$

Coming back to the sum rule and using $\alpha_0(\mu) = \bar{m}_1(\mu)$ we get (5.20).

(2) From (5.19), we have

$$\mathcal{I}^W(\mu) = \mathcal{I}^{\text{GW}}(\tau_{e^{-\mathrm{i}\varphi}}(\mu))\,. \tag{5.28}$$

Therefore, the rate function $\mathcal{I}^W$ has a unique minimum at

$$\mu^\varphi := \tau_{e^{\mathrm{i}\varphi}}(\text{GW}_{\mathtt{g}})\,. \tag{5.29}$$

The Caratheodory transform of the equilibrium measure is ([29] p.86)

$$F(z) = 1 + \mathtt{g}z\,,$$

so that, using (5.29), (5.17) and (5.16), we find the density (5.21). Moreover there is no extra mass since F^φ has no pole on $\mathbb{T}$.

We could also have applied formula (3.2.96) in [29], which states that if $\mu = w(\theta)d\lambda_0(\theta) + d\mu_s(\theta)$, then the density $\tilde{w}$ of μ^φ is given by

$$\tilde{w}(\theta) = \frac{w(\theta)}{|\cos\frac{\varphi}{2} + \mathrm{i}\sin\frac{\varphi}{2}F(e^{\mathrm{i}\theta})|^2}. \tag{5.30}$$

(3) If $m_1(\mu) = e^{\mathrm{i}\varphi}\mathtt{g}/2$, we deduce from (5.20) that

$$\mathcal{I}^{GW}(\mu) = \mathcal{K}(\text{GW}_{\mathtt{g}} \mid \mu) = \mathcal{I}^W(\mu) + \frac{\mathtt{g}^2}{2}(1 - \cos\varphi) \geq \frac{\mathtt{g}^2}{2}(1 - \cos\phi)\,,$$

with equality if and only if $\mu = \mu^\varphi$.

6. Generalizations

In this section, we discuss two possible generalizations of our considerations. The first one concerns the rank-one perturbations of invariant models with general potentials and the second one deals with a matricial version of our results. In each case, for the sake of clarity and to avoid numerous repetitions, we will only treat in details the Hermitian setting.

6.1. General potential.

Additive perturbation. Let V be a convex polynomial potential of even degree $2d$ with positive leading coefficient:

$$V(x) = a_{2d}x^{2d} + \cdots, \quad a_{2d} > 0, \tag{6.1}$$

and suppose that V is convex. Let $P_0^{(n)}$ be the following invariant measure on the set of $n \times n$ Hermitian matrices:

$$\mathbb{P}_0^{(n)}(dX) = \frac{1}{Z_n} \exp\big(-n\mathrm{Tr}\, V(X)\big)dX.$$

Under our assumptions on V, this model has a unique equilibrium measure μ_V, which is the almost-sure limit of the empirical spectral measures. Moreover, μ_V is supported by a single interval $[a_V, b_V]$ and has a density of the form:

$$\mu_V(dx) = \frac{1}{\pi} r(t)\sqrt{(b_V - x)(x - a_V)}\mathbf{1}_{[a_V,b_V]}(x)dx,$$

where r is a polynomial of degree $2d-2$ with nonreal zeros (see for example Proposition 3.1 and Equation (2.8) of [21]).

As in Section 5.1, we are interested in the following additive rank-one perturbation of the model:

$$W := X + \theta e_1 e_1^T, \quad X \sim \mathbb{P}_0^{(n)}.$$

Denoting $\pi = e_1 e_1^T$, the random matrix W has law:

$$\mathbb{P}_\theta^{(n)}(dW) := \exp\big(-n\,[\mathrm{Tr}\, V(W - \theta\pi) - V(W)]\,\big)\mathbb{P}_0^{(n)}(dW). \tag{6.2}$$

Let $\mu_{\mathtt{w}}^{(n)}$ be the spectral measure associated to the pair (W, e_1). In order to compute the distribution of $\mu_{\mathtt{w}}^{(n)}$, we need the following lemma, whose proof is postponed to the end of this section.

Lemma 6.1. *There exists a polynomial Q_V in $2d$ variables such that, for all Hermitian matrix M,*

$$Q_V(\theta, M_{11}, (M^2)_{11}, \ldots, (M^{2d-1})_{11}) = \mathrm{Tr}V(M - \theta\pi) - \mathrm{Tr}V(M).$$

Remark 2. Although a concise formula for Q_{2q} in function of V seems out of reach, let us give two simple examples:

- when $V(x) = x^2$, $Q_V = \theta^2 - 2\theta M_{11}$,
- when $V(x) = x^4$, $Q_V = \theta^4 - 4\theta^3(M^3)_{11} + 4\theta^2(M^2)_{11} + 2\theta^2 M_{11}^2 - 4\theta(M^3)_{11}$.

With the notation of Lemma 6.1, we have that

$$\mathbb{P}_\theta^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu) = \exp\big(-nQ_{2d}(\theta, m_1(\mu_{\mathtt{w}}^{(n)}), \ldots, m_{2d-1}(\mu_{\mathtt{w}}^{(n)}))\big)\mathbb{P}_0^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu),$$

where we recall that $m_i(\mu_{\mathtt{w}}^{(n)})$ stands for the i-th moment of $\mu_{\mathtt{w}}^{(n)}$. We also need the following observation, whose proof is postponed to the end of this section.

Lemma 6.2. *If V is a convex polynomial of even degree,*

$$\sup_n \frac{1}{n} \log \mathbb{E} \exp n\gamma \left(\mathrm{Tr}V(M - \theta\pi) - \mathrm{Tr}V(M)\right) < \infty \tag{6.3}$$

By Lemma 6.2, we may apply Varadhan's Lemma and obtain the following result.

Theorem 6.3. *The sequence of probability measures $(\mu_{\mathtt{w}}^{(n)})_{n\geq 1}$ satisfies a large deviations principle at scale n with good rate function:*

$$\mathcal{I}^W(\mu) = \mathcal{K}(\mu_V \mid \mu) - Q_{2d}(\theta, m_1(\mu_{\mathtt{w}}^{(n)}), \ldots, m_{2d-1}(\mu_{\mathtt{w}}^{(n)})) + \sum_k \mathcal{F}(E_k^{\pm})\,. \tag{6.4}$$

We now turn to the proofs of Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. It is enough to check the assertion when V a monomial $V(x) = cx^r$. The matrix $(M - \theta\pi)^r$ is the sum of 2^r products of elements which are M or $\theta\pi$. Since π is a projection, $\pi^k = \pi$ for every $k \geq 1$, hence the products involved in $(M + \theta\pi)^r - M^r$ are of the form

(1) $\theta^j \pi M^{a_1}\pi ... \pi M^{a_i}\pi$
(2) $\theta^j \pi M^{a_1}\pi ... \pi M^{a_i}$
(3) $\theta^j M^{a_1}\pi ... \pi M^{a_i}$
(4) $\theta^j M^{a_1}\pi ... \pi M^{a_i}\pi\,.$

It is clear that the first expression is exactly $\theta^j (M_{11}^{a_1} \ldots (M^{a_i})_{11}$. The three other ones can be reduced to the first type: since $\mathrm{Tr}AB = \mathrm{Tr}BA$, we can write

- $\mathrm{Tr}(\pi M^{a_1}\pi ... \pi M^{a_i}) = \mathrm{Tr}(\pi^2 M^{a_1}\pi ... \pi M^{a_i}) = \mathrm{Tr}(\pi M^{a_1}\pi ... \pi M^{a_i}\pi)$
- $\mathrm{Tr}(M^{a_1}\pi ... \pi M^{a_i}) = \mathrm{Tr}(M^{a_1+a_i}\pi ... \pi) = \mathrm{Tr}(M^{a_1+a_i}\pi ... \pi^2) = \mathrm{Tr}(\pi(M^{a_1+a_i}\pi ... \pi)$
- $\mathrm{Tr}(M^{a_1}\pi ... \pi M^{a_i}\pi) = \mathrm{Tr}(M^{a_1}\pi ... \pi M^{a_i}\pi^2) = \mathrm{Tr}(\pi M^{a_1}\pi ... \pi M^{a_i}\pi)$

and since $\pi M^k \pi = (M^k)_{11}\pi$, we get the result. □

Proof of Lemma 6.2. Let us denote $\ell := \max\{|\lambda_{\max}|, |\lambda_{\min}|\}$. Combining Lemma 6.1 and the fact that for all $k \geq 0$, $(M^k)_{11} \leq \ell^k$, we deduce that $\mathrm{Tr}\, V(M + \theta\pi) - \mathrm{Tr}\, V(M)$ is bounded by $C\ell^{2d-1}$, for some constant C only depending on V and θ.

Therefore, it is enough to check that $\sup_n n^{-1} \log \mathbb{E} \exp Cn\ell^{2d-1} < \infty$. This fact is a direct consequence of the following rough large deviations estimate : there exists $C' > 0$ such that for every $x > 0$ large enough $\mathbb{P}(\ell > x) \leq e^{-nC'V(x)}$ (see [27] Theorem 11.1.2, a precise rate function is given in [7] Prop. 2.1). The proof is ended recalling that V is given by (6.1). □

Remark 3. If V is analytical:

$$V(z) = \sum_0^\infty c_k z^k,$$

we can formally write

$$\mathrm{Tr}\ \ V(M + \theta\pi) - \mathrm{Tr}\, V(M) = \sum_1^\infty a_k \left(\mathrm{Tr}\,(M + \theta\pi)^k - \mathrm{Tr}\, M^k\right).$$

Using Lemma 6.1, this implies that $\mathrm{Tr}\ \ V(M + \theta\pi) - \mathrm{Tr}\, V(M)$ is a function of all of the moments of the spectral measure associated to the pair (M, e). Of course the problem of uniform exponential integrability seems unreachable in that case.

6.2. **Matricial spectral measures.** Given a matrix M and a r-tuple of unit vectors that are orthogonal $(u_1, \ldots, u_r)$, we define the matricial spectral measure $(\nu_{ij}^M)_{1\leq i,j\leq r}$ as the only matrix of signed measures such that, for all $i, j \in \{1, \ldots, r\}$ and all $z \in \mathbb{C}$, $\Im z > 0$,

$$\langle u_i, (M - zI)^{-1} u_j\rangle = \int \frac{\mathrm{d}\nu_{ij}^M(x)}{x - z}.$$

In particular, for all $k \geq 0$, the following equality holds:

$$\left((M^k)_{ij}\right)_{1\leq i,j\leq r} = \left(\int x^k \mathrm{d}\nu_{ij}^M(x)\right)_{1\leq i,j\leq r}.$$

We will denote by $\mathbf{m}_k = \mathbf{m}_k((\nu_{ij}^M))$ the right-hand side of the above equality. Note that when $r = 1$, we retrieve the previously considered spectral measure associated to the pair (M, u_1). Interestingly, our method also applies to the study of matricial spectral measures of perturbations of the invariant models described in Section 2. In the following, we will always assume that $u_1, \dots, u_r$ are the first r vectors $e_1, \dots, e_r$ of the canonical basis. Analogously to Sections 5.1, 5.2 and 5.3, our results rely on former large deviations principles obtained for the unperturbed models. In order to state them, we first need to introduce the notion of Kullback-Leibler divergence with respect to *quasi-scalar* matricial measures. Let m be a positive measure on $\mathbb{R}$ and let $\boldsymbol{\nu}$ be a matricial measure of size $r \times r$ such that $\boldsymbol{\nu}(dx) = h(x)m(dx) \cdot \mathbf{1}$. Then, for any matricial measure $\boldsymbol{\mu}$ of size $r \times r$, the Kullback-Leibler divergence of $\boldsymbol{\mu}$ with respect to $\boldsymbol{\nu}$ is defined by

$$\mathcal{K}(\boldsymbol{\mu} \mid \boldsymbol{\nu}) = \begin{cases} \displaystyle\int_{\mathbb{R}} \log h(x)\, dm(x) & \text{if } \boldsymbol{\mu} \ll \boldsymbol{\nu}, \\ \infty & \text{otherwise.} \end{cases} \tag{6.5}$$

Finally, we define $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}(\alpha^-, \alpha^+)$ the set of all bounded matricial measures $\boldsymbol{\mu}$ of size $r \times r$ such that

(i) $\mathrm{supp}(\boldsymbol{\mu}) = J \cup \{E_i^-\}_{i=1}^{N^-} \cup \{E_i^+\}_{i=1}^{N^+}$, where $J \subset I = [\alpha^-, \alpha^+]$, $N^-, N^+ \in \mathbb{N} \cup \{\infty\}$ and

$$E_1^- < E_2^- < \cdots < \alpha^- \quad \text{and} \quad E_1^+ > E_2^+ > \cdots > \alpha^+.$$

(ii) If N^- (resp. N^+) is infinite, then E_j^- converges towards α^- (resp. λ_j^+ converges to α^+).

Such a matricial measure $\boldsymbol{\mu}$ can always be written as

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{|I} + \sum_{i=1}^{N^+} \boldsymbol{\Gamma}_i^+ \delta_{E_i^+} + \sum_{i=1}^{N^-} \boldsymbol{\Gamma}_i^- \delta_{E_i^-}, \tag{6.6}$$

for some $r \times r$ matrices $\boldsymbol{\Gamma}_i^{\pm}$. We also introduce

$$\boldsymbol{\mathcal{S}}_1 = \boldsymbol{\mathcal{S}}_1(\alpha^-, \alpha^+) := \{\mu \in \boldsymbol{\mathcal{S}} |\, \mu(\mathbb{R}) = \mathbf{1}\},$$

and endow $\boldsymbol{\mathcal{S}}_1$ with the weak topology and the corresponding Borel σ-algebra.

In the unitary case, there is a corresponding framework. We omit to give details for simplicity.

The Hermitian case. For all $n \geq r$, let X_n be a GUE(n) random matrix, meaning that X has law $\mathbb{P}_0^{(n)}$. Let also A_n be a deterministic Hermitian matrix having all of its entries equal to zero except for the $r \times r$ top-left block which is given by some Hermitian matrix $\boldsymbol{\Theta}$. We are interested in the matricial spectral measure of the deformed matrix:

$$W_n := X_n + A_n.$$

Note that the law $\mathbb{P}_{\boldsymbol{\Theta}}^{(n)}$ of W_n is given by:

$$\mathbb{P}_{\boldsymbol{\Theta}}^{(n)}(dW) = \frac{1}{\mathcal{Z}_0^{(n)}} \exp\left(-\frac{n}{2}\mathrm{Tr}(W - A_n)(W - A_n)^\star\right) dW.$$

Let ${\boldsymbol{\mu}_{\mathtt{w}}}^{(n)}$ be the matricial spectral measure associated to W_n and the r-tuple $(e_1, \dots, e_r)$. Since

$$\begin{aligned}\mathrm{Tr}(W - A_n)(W - A_n)^\star &= \mathrm{Tr}(WW^\star) - 2\mathrm{Tr}(A_n W) + \mathrm{Tr}(A_n A_n^\star) \\ &= \mathrm{Tr}(WW^\star) - 2\mathrm{Tr}(\boldsymbol{\Theta}\mathbf{m}_1) + \mathrm{Tr}\left(\boldsymbol{\Theta}\boldsymbol{\Theta}^\star\right),\end{aligned}$$

we deduce that

$$\mathbb{P}_{\boldsymbol{\Theta}}^{(n)}({\boldsymbol{\mu}_{\mathtt{w}}}^{(n)} \in d\boldsymbol{\mu}) = \frac{\exp n\Psi(\boldsymbol{\mu})}{\mathbb{E}_0[\exp n\Psi({\boldsymbol{\mu}_{\mathtt{w}}}^{(n)})]}\mathbb{P}_{\mathbf{0}}^{(n)}({\boldsymbol{\mu}_{\mathtt{w}}}^{(n)} \in d\boldsymbol{\mu}), \tag{6.7}$$

where $\mathbf{0}$ is the $r \times r$ matrix having all its coefficients equal to zero and where

$$\Psi(\boldsymbol{\mu}) = \mathrm{Tr}\left(\boldsymbol{\Theta}\mathbf{m}_1(\boldsymbol{\mu})\right).$$

Under $\mathbb{P}_{\mathbf{0}}^{(n)}$, it is known (see for example [15]) that the sequence $({\boldsymbol{\mu}_{\mathtt{w}}}^{(n)})_{n\geq r}$ satisfies a large deviations principle at speed n and with good rate function

$$\mathcal{I}^X(\boldsymbol{\mu}) = \begin{cases} \mathcal{K}(SC\cdot\mathbf{1};\boldsymbol{\mu}) + \sum\limits_{k\geq 1}\mathcal{F}_H(E_k^\pm) & \text{if } \boldsymbol{\mu} \in \boldsymbol{\mathcal{S}}_1(-2,2), \\ \infty & \text{otherwise.}\end{cases}$$

Besides, note that for every $\gamma > 0$,

$$\frac{1}{n}\log\mathbb{E}_{\mathbf{0}}\left[\exp\mathrm{Tr}\left(\boldsymbol{\Theta}\mathbf{m}_1\right)\right] = \frac{\gamma^2}{2}\mathrm{Tr}\left(\boldsymbol{\Theta}\boldsymbol{\Theta}^\star\right).$$

Therefore, applying Varadhan's Lemma to (6.7), we obtain the following analog of Theorem 5.1.

Theorem 6.4. *The sequence $({\boldsymbol{\mu}_{\mathtt{w}}}^{(n)})_{n\geq r}$ satisfies a large deviations principle at speed n and with good rate function $\mathcal{I}^W$ given by*

$$\mathcal{I}^W(\boldsymbol{\mu}) = \begin{cases}\mathcal{K}(SC\cdot\mathbf{1}\mid\boldsymbol{\mu}) + \sum\limits_{k\geq 1}\mathcal{F}_H(E_k^\pm) - \mathrm{Tr}(\boldsymbol{\Theta}\mathbf{m}_1) + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\Theta}\boldsymbol{\Theta}^\star) & \text{if } \boldsymbol{\mu} \in \boldsymbol{\mathcal{S}}_1(-2,2), \\ \infty & \text{otherwise.}\end{cases}$$

Let us finally describe the unique minimizer of $\mathcal{I}^W$. First, we claim that, as in the scalar case described in Section 3, there exists a one-to-one correpsondence between matricial measures $\boldsymbol{\mu}$ and sequences of $r\times r$ matrices $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ such that the matrices B_i's are Hermitian positive definite. Using the matricial sum rule (Th. 2.1 in [15]), the good rate function can be rewritten, when $\boldsymbol{\mu} \in \boldsymbol{\mathcal{S}}_1(-2,2)$:

$$\mathcal{I}^W(\boldsymbol{\mu}) = \frac{1}{2}\mathrm{Tr}(B_1 - \boldsymbol{\Theta})(B_1 - \boldsymbol{\Theta})^\star + \frac{1}{2}\sum_{n\geq 2}\mathrm{Tr}B_n B_n^\star + \sum_{n\geq 1} G(A_n A_n^\star).$$

The unique minimizer $\boldsymbol{\mu}_{SC,\boldsymbol{\Theta}} = \mathrm{argmin}\,\mathcal{I}^W$ can therefore be described by its matricial Jacobi coefficients:

$$\mathrm{Jac}(\boldsymbol{\mu}_{SC,\boldsymbol{\Theta}}) = \begin{pmatrix}\boldsymbol{\Theta}, & \mathbf{0}, & \mathbf{0}, & \cdots \\ \mathbf{1}, & \mathbf{1}, & \mathbf{1}, & \cdots\end{pmatrix}.$$

In order to obtain an explicit formula, we compute the matricial Stieltjes transform of $\boldsymbol{\mu}_{SC,\boldsymbol{\Theta}}$, which is defined by

$$\mathbf{G}(z) := \int\frac{\mathrm{d}\boldsymbol{\mu}_{SC,\boldsymbol{\Theta}}(x)}{x - z\mathbf{1}}.$$

By [32, Theorem 4.3.3], it satisfies the following equation:

$$\mathbf{G}(z) = \frac{1}{\boldsymbol{\Theta} - z\mathbf{1} - G_{SC}(z)\mathbf{1}}, \tag{6.8}$$

where $G_{SC}(z) = \frac{1}{2}(z - \sqrt{z^2-4})$ is the Stieltjes transform of the semi-circle law. Therefore:

$$\mathbf{G}(z) = \frac{(z-\sqrt{z^2-4})\mathbf{1} - 2\mathbf{\Theta}}{2\left(\mathbf{\Theta}\mathbf{\Theta}^\star + \mathbf{1} - z\mathbf{\Theta}\right)}.$$

Since the absolutely continuous part of $\boldsymbol{\mu}_{SC,\mathbf{\Theta}}$ is given by

$$\frac{d\boldsymbol{\mu}_{SC,\mathbf{\Theta}}(x)}{dx} = \lim_{t\to 0^+} \frac{1}{\pi}\Im \mathbf{G}(x+it),$$

it is easy to deduce that

$$\frac{d\boldsymbol{\mu}_{SC,\mathbf{\Theta}}(x)}{dx} = \frac{\sqrt{(4-x^2)_+}}{2\pi}\left(\mathbf{\Theta}\mathbf{\Theta}^\star + \mathbf{1} - x\mathbf{\Theta}\right)^{-1}.$$

Moreover, $\boldsymbol{\mu}_{SC,\mathbf{\Theta}}$ has an atom at each pole of $\mathbf{G}$ and the mass of this atom is the corresponding residue. Thanks to (6.8), the poles of $\mathbf{G}$ corresponds to the reals x such that $\det(\mathbf{\Theta} - x\mathbf{1} - G_{SC}(x)\mathbf{1}) = 0$. For simplicity, let us assume from now on that Θ has distinct eigenvalues $\theta_1, \ldots, \theta_r$, the adaptation in the general case being straightforward. Let U be the matrix whose columns are the eigenvectors of $\mathbf{\Theta}$. Then, $\mathbf{\Theta} = UDU^\star$ with $D = \mathrm{Diag}(\theta_1, \ldots, \theta_r)$, and we deduce that

$$\mathbf{G}(z) = U\frac{1}{D - (z + G_{SC}(z))\mathbf{1}}U^\star.$$

We now use the following well-known fact about the subordination function $\omega(z) = z + G_{SC}(z)$:

- if $|\theta| \le 1$, there is no real x such that $\omega(x) = \theta$;
- if $|\theta| > 1$, there exists exactly one real $x_\theta = \theta + \frac{1}{\theta}$ such that $|x_\theta| > 2$ and $\omega(x_\theta) = \theta$. Moreover, $1/\omega'(x_\theta) = 1 - \frac{1}{\theta^2}$.

Therefore, the poles of $\mathbf{G}$ are in one-to-one correspondence with the eigenvalues θ_i of $\mathbf{\Theta}$ satisfying $|\theta_i| > 1$, and each of this pole has a residue given by $U(1 - \frac{1}{\theta_i^2})e_i e_i^T U^\star$. Hence, we have proved that:

$$\begin{aligned}\boldsymbol{\mu}_{SC,\mathbf{\Theta}}(dx) &= \frac{\sqrt{(4-x^2)_+}}{2\pi}\left(\mathbf{\Theta}\mathbf{\Theta}^\star + \mathbf{1} - x\mathbf{\Theta}\right)^{-1}dx \\ &\quad + \sum_{i=1}^r \left(1 - \frac{1}{\theta_i^2}\right) U e_i e_i^T U^\star \mathbf{1}_{|\theta_i|>1}\delta_{\theta_i + \frac{1}{\theta_i}}(dx). \qquad (6.9)\end{aligned}$$

It can also be written as follows:

$$\boldsymbol{\mu}_{SC,\mathbf{\Theta}} = U\,\mathrm{Diag}(\mu_{\mathrm{SC},\theta_1}, \cdots, \mu_{\mathrm{SC},\theta_r})U^*.$$

Application. The above computation is particularly simple when $\mathbf{\Theta}$ is proportional to a projection, i.e. $\mathbf{\Theta} = \theta\boldsymbol{R}$ with $\boldsymbol{R}^2 = \boldsymbol{R}$. Then

$$\begin{aligned}\mathbf{\Theta}^2 + \mathbf{1} - z\mathbf{\Theta} &= \mathbf{1} - (\theta z - \theta^2)\boldsymbol{R} \\ (\mathbf{\Theta}^2 + \mathbf{1} - z\mathbf{\Theta})^{-1} &= \mathbf{1} + \sum_1^\infty (\theta z - \theta^2)^n \boldsymbol{R}^n = \mathbf{1} + \boldsymbol{R}\sum_1^\infty (\theta z - \theta^2)^n \\ &= \mathbf{1} + \left((1 - \theta z + \theta^2)^{-1} - 1\right)\boldsymbol{R}. \qquad (6.10)\end{aligned}$$

In particular, let us consider the following problem. Let u such that $|\langle u, e\rangle| \neq 1$. We look for the spectral measure $\mu^{u,e}$ of the pair $(X + \theta uu^*, e)$. To build a basis, we set $e_1 = e$, $w = u - \langle u, e\rangle e$, $e_2 = \frac{w}{\|w\|}$ and complete $e_3, \cdots, e_n$. We have

$$u = (\cos\varphi, \sin\varphi)^T$$
$$\boldsymbol{R} = uu^* = \begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}\ ,\ U = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

Our (scalar) spectral measure $\mu^{u,e}$ is exactly $\left(\boldsymbol{\mu}_{\mathrm{SC},\boldsymbol{\Theta}}\right)_{11}$ and then, from (6.9) and (6.10)

$$\mu^{u,e} = (\sin^2\varphi)\,\mathrm{SC} + (\cos^2\varphi)\mu_{\mathrm{SC},\theta}\,. \tag{6.11}$$

In short, we just proved that as $n \to \infty$ the spectral measure $\mu^{(n)}$ of the pair $(X_n + \theta u_n u_n^*, e_n)$ where e_n and u_n are two unit vectors such that $\mathbf{u}_n^* e_n = \cos\varphi$ converges almost surely to $\mu^{u,e}$. Of course, by the contraction principle, $\boldsymbol{\mu}^{(n)}$ satisfies the LDP at speed n with rate function

$$\mathcal{I}^{u,e}(\mu) = \inf\{\mathcal{I}^W(\boldsymbol{\mu}); (\boldsymbol{\mu})_{11} = \mu\}\,.$$

where $\mathcal{I}^W$ was defined in Theorem 6.4 but we didn't find anl expression of this rate function.

The Gross-Witten case. The role of $e^{\mathrm{i}\varphi}$ is now played by a unitary $r\times r$ operator. In the sequel, we will omit the subscript n to simpifly the notation. As in (4.5.10) in [29] we consider

$$W = UQ\ ,\ Q = 1 + (\Lambda - 1)P\,, \tag{6.12}$$

where P is the projection on $\mathcal{H}_r = \mathrm{Vect}\ \{e_1, \dots, e_r\}$ and Λ is a unitary operator acting on $\mathcal{H}_r$. Notice that

$$Q^{-1} = Q^* = 1 + (\Lambda^* - 1)P\,.$$

In other words,

$$Q = \Lambda \oplus I_{n-r}\ ,\ Q^{-1} = Q^* = \Lambda^* \oplus I_{n-r}\,.$$

If μ is the spectral measure of the pair $(\mu; e_1, \dots, e_r)$, let us denote by $\tau_\Lambda\mu$ the spectral measure of the pair $(W; e_1, \dots, e_r)$. We have the matricial version of (5.17) (Theorem 4.5.6 in [29])

$$F_\Lambda = \left[(\mathbf{1}+\Lambda) - F(\mathbf{1}-\Lambda)\right]^{-1}\left[-(\mathbf{1}-\Lambda) + F(\mathbf{1}+\Lambda)\right]\,.$$

which gives, via the Schur recursion

$$\boldsymbol{\alpha}_k(\tau_\Lambda\mu) = \Lambda^*\boldsymbol{\alpha}_k(\mu)\ ,\ (k \geq 0)\,. \tag{6.13}$$

To compute the distribution of W, let us denote by $W_r^\uparrow$ the $r\times r$ upper left corner of W and by $W_{n-r}^\downarrow$ the $(n-r)\times(n-r)$ lower right corner of W so that

$$\mathbb{P}_\Lambda^{(n)}(dW) = \frac{1}{\mathcal{Z}_0^{(n)}}\exp\frac{n\mathsf{g}}{2}\mathrm{Tr}\big(WQ^{-1} + (WQ^{-1})^*\big)\,dW\,. \tag{6.14}$$

Let $\boldsymbol{\mu}_{\mathtt{w}}^{(n)}$ be the matricial spectral measure associated to W and the r-tuple $(e_1, \dots, e_r)$. Since

$$\mathrm{Tr}(WQ^{-1}) = \mathrm{Tr}\,W + \mathrm{Tr}\,\left(W_r^\uparrow(\Lambda^* - \mathbf{1})\right)$$
$$\mathrm{Tr}\left(WQ^{-1} + (WQ^{-1})^*\right) = \mathrm{Tr}\,(W + W^*) + 2\Re\mathrm{Tr}\,\left(W_r^\uparrow(\Lambda^* - \mathbf{1})\right) \tag{6.15}$$

so that

$$\mathbb{P}_\Lambda^{(n)}(dW) = \exp n\mathsf{g}\Re\mathrm{Tr}\,\left(W_r^\uparrow(\Lambda^* - \mathbf{1})\right)\ \mathbb{P}^{(n)}(dV) \tag{6.16}$$

and since $W_r^{\uparrow} = \boldsymbol{\alpha}_0^* = \boldsymbol{m}_1(\mu_{\mathtt{w}}^{(n)})$, we get

$$\mathbb{P}_\Lambda^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu) = \exp n\mathtt{g}\Re\mathrm{Tr}\ (\boldsymbol{m}_1(\mu)(\Lambda^* - \mathbf{1}))\ \mathbb{P}_{\mathbf{1}}^{(n)}(\mu_{\mathtt{w}}^{(n)} \in d\mu)\,. \tag{6.17}$$

Under $\mathbb{P}_{\mathbf{1}}^{(n)}$, it is known ([14]) that the sequence $(\boldsymbol{\mu}_{\mathtt{w}}^{(n)})_{n\geq r}$, satisfies an LDP at speed n. If $|\mathtt{g}| \leq 1$, the rate function is

$$\mathcal{I}^{\mathrm{GW}}(\boldsymbol{\mu}) = \mathcal{K}(\mathrm{GW}_{\mathtt{g}} \cdot \mathbf{1} \mid \boldsymbol{\mu})\,. \tag{6.18}$$

The matrix measure $\mathrm{GW}_{\mathtt{g}} \cdot \mathbf{1}$ is the unique minimum of $\mathcal{I}$.

This allows to obtain the following analog of Theorem 5.3.

Theorem 6.5. *The sequence $(\boldsymbol{\mu}_{\mathtt{w}}^{(n)})_{n\geq r}$ satisfies an LDP at speed n and good rate function*

$$\mathcal{I}^W(\boldsymbol{\mu}) = \mathcal{I}^{GW}(\boldsymbol{\mu}) - \mathtt{g}\Re\mathrm{Tr}\ (\boldsymbol{m}_1(\mu)(\Lambda^* - \mathbf{1}))\ . \tag{6.19}$$

There is a matrix version of the method to recover the measure (Prop. 3.16 in [10] and Lemma 7.1 in [6]). From (6.13), it is then straightforward to state that if $d\boldsymbol{\mu}(\theta) = w(\theta)\cdot\mathbf{1}\ d\lambda_0(\theta) + d\boldsymbol{\mu}_s(\theta)$, then $\tau_\Lambda(\boldsymbol{\mu}\cdot\mathbf{1})$ has for density

$$w^\Lambda(\theta) = 4w(\theta)\,|\mathbf{1} + \Lambda + F(\theta)(\mathbf{1} - \Lambda)|^{-2} \tag{6.20}$$

where $|A|^2 = AA^*$ (analog of (5.30). Notice that if $|\mathtt{g}| \leq 1$ there is no extra mass. From (6.13) we have

$$\mathbb{P}_\Lambda^{(n)}(\boldsymbol{\mu}_{\mathtt{w}}^{(n)} \in d\boldsymbol{\mu}) = \mathbb{P}_{\mathbf{1}}^{(n)}\left(\boldsymbol{\mu}_{\mathtt{w}}^{(n)} \in d(\tau_{\Lambda^*}\boldsymbol{\mu})\right)\,. \tag{6.21}$$

Under $\mathbb{GW}_{\mathtt{g}}^{(n)}$, the rate function for the LDP is $\mathcal{K}(\mathrm{GW}_{\mathtt{g}} \cdot \mathbf{1} \mid \boldsymbol{\mu})$. A pushforward of this LDP gives

$$\mathcal{I}^W(\boldsymbol{\mu}) = \mathcal{K}(\mathrm{GW}_{\mathtt{g}} \cdot \mathbf{1} \mid \tau_\Lambda\boldsymbol{\mu})\,. \tag{6.22}$$

It is then clear that $\mathcal{I}^W$ reaches his unique minimum at $\tau_{\Lambda*}(\mathrm{GW}_{\mathtt{g}} \cdot \mathbf{1})$.

Remark 4. We didn't give an alternate proof of the LDP. Actually we could have used the matrix version of the sum rule (4.15) proved recently by analytic methods in [28].

$$\mathcal{K}(\mathrm{GW}_{-\mathtt{g}} \cdot \mathbf{1} \mid \boldsymbol{\mu}) = \mathcal{K}(\mathrm{GW}_{-\mathtt{g}} \mid \lambda_0) - \sum_0^\infty \log\det(\mathbf{1} - \alpha_k\alpha_k^*) + \mathtt{g}T(\alpha_0, \alpha_1, \cdots)\,. \tag{6.23}$$

with

$$\begin{aligned} T(\alpha_0, \alpha_1, \cdots) = \Re\mathrm{Tr}\,\alpha_0 + \frac{1}{2}\mathrm{Tr}\,\alpha_0\alpha_0^* \\ + \frac{1}{2}\sum_0^\infty \mathrm{Tr}\,(\alpha_k - \alpha_{k+1})(\alpha_k^* - \alpha_{k+1}^*) - \sum_0^\infty \mathrm{Tr}\,\alpha_k\alpha_k^*\,. \end{aligned} \tag{6.24}$$

7. Appendix

We use the affine transformation $T_{\alpha,\beta}$ corresponding to the change of variable $x = \alpha y + \beta$.

7.1. **A technical result.** The first lemma is elementary. We give its proof for the sake of completeness.

Lemma 7.1. *If*

$$\mathrm{Jac}(\mu) = \begin{pmatrix} b_1, & b_2, & \cdots \\ a_1, & a_2, & \cdots \end{pmatrix} \tag{7.1}$$

then

$$\mathrm{Jac}\left(T_{r,s}(\mu)\right) = \begin{pmatrix} \tilde{b}_1, & \tilde{b}_2, & \cdots \\ \tilde{a}_1, & \tilde{a}_2, & \cdots \end{pmatrix} \text{ with } \tilde{a}_k = \frac{a_k}{|r|} \ , \ \tilde{b}_k = \frac{b_k - s}{r} \, . \tag{7.2}$$

Proof. If J be the Jacobi matrix associated with μ

$$\langle e, (J-z)^{-1} e\rangle = \int \frac{d\mu(x)}{x-z}$$

hence

$$\int \frac{dT_{r,s}(\mu)(y)}{y-z} = \int \frac{d\mu(x)}{r^{-1}(x-s)-z} = \langle e, (r^{-1}(J-s)-z)^{-1}\rangle$$

hence if $r > 0$ the Jacobi matrix associated to $T_{r,s}(\mu)$ is $\tilde{J} = r^{-1}(J-s)$.

If $r = -1, s = 0$, the tridiagonal operator $-J$ admits $T_{-1,0}$ as its spectral measure, but $-J$ is not Jacobi. A change of basis $e_k \mapsto \tilde{e}_k = (-1)^{k-1} e_k$ gives the true Jacobi with $\tilde{b}_k = \langle \tilde{e}_k, (-J)\tilde{e}_k\rangle = -b_k$ and $\tilde{a}_k = \langle \tilde{e}_{k+1}, (-J)\tilde{e}_k\rangle = a_k$.

7.2. **Free Meixner distributions.** From [2], we know[3] that the normalized free Meixner distributions $\mu_{b,c}$ are probability measures on $\mathbb{R}$ with Jacobi parameter sequences

$$\mathrm{Jac}(\mu_{b,c}) = \begin{pmatrix} 0, & b, & b, & \cdots \\ 1, & \sqrt{1+c}, & \sqrt{1+c}, & \cdots \end{pmatrix} \tag{7.3}$$

$b \in \mathbb{R}, c > -1$. The first line corresponds to the b's (diagonal terms) and the second to the a's (subdiagonal terms). The corresponding probability measure is

$$\mu_{b,c}(dx) := \frac{1}{2\pi} \cdot \frac{\sqrt{4(1+c)-(x-b)^2}}{1+bx+cx^2} dx + p_1\delta_{x_1} + p_2\delta_{x_2} \, , \tag{7.4}$$

where x_1 and x_2 are real roots of $1+bx+cx^2 = 0$ (if there exist(s)) and $p_1, p_2 \in [0,1)$. The mean is 0 and the variance is 1.

The case $b = c = 0$ and $p_1 = p_2 = 0$ is just SC also called **"free Gaussian"**. In order to compare $\mu_{b,c}$ with SC, we transform the support into $[-2,2]$ and set

$$\tilde{\mu}_{b,c}(dy) := T_{\sqrt{1+c},b}(\mu_{b,c})(dy) := \frac{1}{2\pi} \cdot \frac{\sqrt{4-y^2}}{cy^2+\alpha y+\beta} dy + p_1\delta_{y_1} + p_2\delta_{y_2} \tag{7.5}$$

with

$$\mathrm{Jac}\left(\tilde{\mu}_{b,c}\right) = \begin{pmatrix} -b/\sqrt{1+c}, & 0, & 0, & \cdots \\ 1/\sqrt{1+c}, & 1, & 1, & \cdots \end{pmatrix} . \tag{7.6}$$

Apart from SC there are only 5 situations.

[3] Be careful, the author considered the sequence $\{a_n^2, b_n\}$ as Jacobi coefficients.

(1) $c = 0$, ($b \neq 0$).

$$\mu_{b,0}(dx) = \frac{1}{2\pi} \cdot \frac{\sqrt{4-(x-b)^2}}{1+bx} + (1-b^{-2})^+ \delta_{-b^{-1}} \tag{7.7}$$

$$T_{1,b}(\mu_{b,0})(dy) = \frac{1}{2\pi} \frac{\sqrt{4-y^2}}{(1+b^2)+by} dy + (1-b^{-2})^+ \, \delta_{-b-b^{-1}} \,. \tag{7.8}$$

It is a variant of MP, called also **"free Poisson"**. Indeed,

$$T_{b,1}(\mu_{b,0})(dy) = \frac{1}{2\pi b^2} \cdot \frac{\sqrt{((1+b)^2-y)(y-(1-b)^2)}}{y} dy + (1-b^{-2})^+ \delta_0$$

(2) $c \neq 0$

(a) $-1 < c < 0$, it is called **"free binomial"**, the denominator has two real roots. For instance, when $b = 0$ we get the measure

$$\mu_{0,c}(dx) = \frac{1}{2\pi} \cdot \frac{\sqrt{4(1+c)-x^2}}{1+cx^2} dx + p \left(\delta_{-1/\sqrt{(-c)}} + \delta_{1/\sqrt{(-c)}} \right) , \tag{7.9}$$

with $p = \left(1 + \frac{1}{2c}\right)^+$,

$$T_{\sqrt{1+c},0}(\mu_{0,c})(dy) = \frac{1}{2\pi} \cdot \frac{\sqrt{4-y^2}}{(1+c)^{-1}+cy^2} dy + p \left(\delta_{-1/\sqrt{-c(1+c)}} + \delta_{1/\sqrt{-c(1+c)}} \right) . \tag{7.10}$$

Notice that the variance is $\sigma^2 = 1/(1+c) > 1$. There are masses if and only if $c \in (-1, -1/2)$.
Up to a affine transform, this distribution is of the KMK type. In other words it is the equilibrium measure when the potential is $-n\kappa_2 \log x - n\kappa_1 \log(1-x)$ (see Appendix)

(b) $c > 0, b^2 - 4c < 0$, for instance with $b = 0$. We get

$$\mu_{0,c}(dx) = \frac{1}{2\pi} \cdot \frac{\sqrt{4(1+c)-x^2}}{1+cx^2} dx \tag{7.11}$$

(without any atoms). It is called **"free hyperbolic tangent"** or **"free Meixner type"**, and

$$T_{\sqrt{1+c},0}(\mu_{0,c})(dy) = \frac{1}{2\pi} \cdot \frac{\sqrt{4-y^2}}{(1+c)^{-1}+cy^2} dy \,.$$

Notice that the variance is $\sigma^2 = 1/(1+c) < 1$. Up to a scaling, this distribution can be obtained by Cayley transform from the Hua-Pickrell distribution. In other words it is the equilibrium measure when the potential is $n \log(1+x^2)$ (see [14]).

(c) $b^2 = 4c$, one double root $x = -2/b$, the measure is

$$\mu_{b,b^2/4}(dx) = \frac{1}{2\pi} \cdot \frac{\sqrt{4+2bx-x^2}}{\left(1+\frac{bx}{2}\right)^2} dx \,,$$

It is sometimes called **"free Gamma type"** and

$$T_{\sqrt{1+\frac{b^2}{4}},b}(\mu_{b,b^2/4})(dy) = \frac{1}{2\pi} \cdot \frac{\sqrt{4-y^2}}{\left(\frac{b}{2}y + \frac{b^2+2}{\sqrt{b^2+4}}\right)^2} dy \,. \tag{7.12}$$

(d) $c > 0, b^2 - 4c > 0$, it is called **"free Pascal"**, the denominator in (7.4) has two real roots

$$x_\pm = -\frac{b}{2c} \pm \operatorname{sgn} b \frac{\sqrt{b^2 - 2c}}{2c}$$

and there is a mass $p = \left(1 - \frac{|b| - \sqrt{b^2-4c}}{2c\sqrt{b^2-4c}}\right)^+$ at x_+, and

$$T_{\sqrt{1+c},b}(\mu_{b,c})(dx) = \frac{1}{2\pi} \cdot \frac{\sqrt{4 - y^2}}{c(y - y_+)(y - y_-)} + p\delta_{y_+}\,, \tag{7.13}$$

where $y_+ = \frac{x_+ - b}{\sqrt{1+c}}$.

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