Spectral Measures of Spiked Random Matrices

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Additive Perturbation

For all $n \geq 1$, we consider the following $n \times n$ random matrix:

$$W_n = \frac{1}{\sqrt{n}} X_n + \theta e_1 e_1^T,$$

where $\theta \geq 0$ and

- X_n is a symmetric $n \times n$ random matrix with i.i.d. entries that are centered and reduced;
- e_1 is the first vector of the canonical basis.

Let $\lambda_1(W_n) \geq \cdots \geq \lambda_n(W_n)$ be the eigenvalues of W_n and $\phi_1(W_n), \ldots, \phi_n(W_n)$ the associated eigenvectors.

Multiplicative Perturbation

For all $n \geq 1$, we consider the following $n \times n$ random matrix:

$$S_n = \frac{1}{n} \Sigma_n^{1/2} X_n X_n^T \Sigma_n^{1/2},$$

where $\theta \geq 1$ and

- X_n is a $n \times \alpha n$ ($\alpha \geq 1$) random matrix with i.d.d. entries that are centered and reduced;
- $\Sigma_n = \mathrm{Diag}(\theta, 1, \ldots, 1)$ is of size $n \times n$.

Let $\lambda_1(S_n) \geq \cdots \geq \lambda_n(S_n)$ be the eigenvalues of S_n and $\phi_1(S_n), \ldots, \phi_n(S_n)$ the associated eigenvectors.

Spectral Measures in Direction e_1

Heuristically, when $\theta \gg 1$, it should be close to an eigenvalue of W_n (resp. S_n) and its eigenvector should be close to e_1. In order to reveal the influence of θ at a macroscopical level, one should therefore look at a statistic which *projects* the spectrum of W_n (resp. S_n) onto e_1. This is precisely the definition of the spectral measures in direction e_1:

$$\mu_{(W_n,e_1)} := \sum_{i=1}^n |\langle \phi_i(W_n), e_1 \rangle|^2 \, \delta_{\lambda_i(W_n)} \quad \text{and} \quad \mu_{(S_n,e_1)} := \sum_{i=1}^n |\langle \phi_i(S_n), e_1 \rangle|^2 \, \delta_{\lambda_i(S_n)}.$$

Their Stieltjes transforms are given by $\langle e_1, (W_n - z)^{-1} e_1 \rangle$ and $\langle e_1, (S_n - z)^{-1} e_1 \rangle$. These quantities have been carefully analyzed, the most recent results being the local laws obtained by Knowles and Yin in [1]. For fixed z, it implies the pointwise convergence of the Stieltjes transforms. Interestingly, the associated limiting probability measures are explicit, given respectively by:

$$\mu_{sc,\theta}(\mathrm{d}x) = \frac{\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}\mathrm{d}x}{2\pi(\theta^2 - \theta x + 1)} + \mathbf{1}_{|\theta|>1}\left(1 - \frac{1}{\theta^2}\right)\delta_{\theta+\frac{1}{\theta}}(\mathrm{d}x) \quad \text{and} \quad \mu_{\alpha,\theta}(\mathrm{d}x) = \frac{\theta\sqrt{(b-x)(x-a)}\mathbf{1}_{(a,b)}(x)\mathrm{d}x}{2\pi x(x(1-\theta)+\theta(\alpha\theta-\alpha+1))} + d_{\alpha,\theta}\mathbf{1}_{|\theta-1|>\frac{1}{\sqrt{\alpha}}}\delta_{x_{\alpha,\theta}}(\mathrm{d}x),$$

where $a, b = (1 \mp \sqrt{\alpha})^2$ and where $x_{\alpha,\theta}$ and $d_{\alpha,\theta}$ are explicit constants.

Additive Model: Outliers

When $\theta > 1$, we recover the following classical convergence:

- $\lambda_1(W_n) \xrightarrow{\mathbb{P}} \theta + 1/\theta$;
- $|\langle \phi_1(W_n), e_1 \rangle| \xrightarrow{\mathbb{P}} \sqrt{1 - 1/\theta^2}$.

Multiplicative Model: Outliers

When $\theta > 1 + 1/\sqrt{\alpha}$, we recover the following classical convergence:

- $\lambda_1(S_n) \xrightarrow{\mathbb{P}} x_{\alpha,\theta}$;
- $|\langle \phi_1(W_n), e_1 \rangle| \xrightarrow{\mathbb{P}} \sqrt{d_{\alpha,\theta}}$.

Additive model: overlaps

Let $x \in (-2,2)$ and $\varepsilon > 0$. Then for any sequence $1 \gg \varepsilon_n \gg n^{-1/2+\varepsilon}$, denoting

$$\mathcal{I}_{\varepsilon_n}(x) = \{i : |\lambda_i(W_n) - x| \leq \varepsilon_n\},$$

it holds that

$$\frac{n}{|\mathcal{I}_{\varepsilon_n}(x)|} \sum_{i \in \mathcal{I}_{\varepsilon_n}(x)} |\langle \phi_i(W_n), e_1 \rangle|^2 \xrightarrow{\mathbb{P}} \frac{1}{\theta^2 - \theta x + 1}. \qquad (1)$$

Simulations for $n = 4000$, $\theta = 2$:

Multiplicative model: overlaps

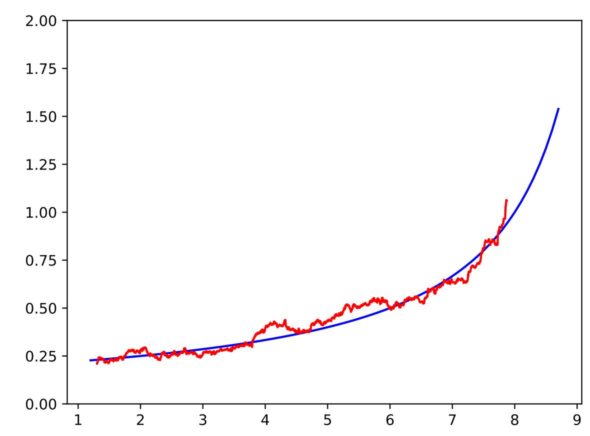
Let $x \in (a,b)$ and $\varepsilon > 0$. Then, for any sequence $1 \gg \varepsilon_n \gg n^{-1/2+\varepsilon}$, denoting

$$\mathcal{I}_{\varepsilon_n}(x) = \{i : |\lambda_i(W_n) - x| \leq \varepsilon_n\},$$

it holds that

$$\frac{n}{|\mathcal{I}_{\varepsilon_n}(x)|} \sum_{i \in \mathcal{I}_{\varepsilon_n}(x)} |\langle \phi_i(W_n), e_1 \rangle|^2 \xrightarrow{\mathbb{P}} \frac{\theta}{x(1-\theta) + \theta(\alpha\theta - \alpha + 1)}. \qquad (2)$$

Simulations for $n = 2000$, $\alpha = 4$, $\theta = 2$:



Sketch of proofs

We focus on the Wigner case as the arguments are the same in the Wishart setting.

- **Outliers.** The largest eigenvalue $\lambda_1(n^{-1/2}X_n)$ converges in probability towards 2. When $\theta > 0$, the eigenvalues of W_n interlace with those of $n^{-1/2}X_n$. Therefore, only $\lambda_1(W_n)$ may be asymptotically larger than 2. It implies that the atom of $\mu_{sc,\theta}$ at $\theta + 1/\theta$, which exists whenever $\theta > 1$, is the limit of $\lambda_1(W_n)$ in probability. Moreover, its mass is the limit of the projection of $\phi_1(W_n)$ onto the direction of the spike.

- **Overlaps.** Let $x \in (-2,2)$, define $I_{\varepsilon_n}(x) := [x - \varepsilon_n, x + \varepsilon_n]$ and denote $f_{sc,\theta}$ the density of $\mu_{sc,\theta}$. The idea is to estimate $\mu_{(W_n,e_1)}(I_{\varepsilon_n}(x))$ in two different ways. Firstly, $\mu_{(W_n,e_1)}(I_{\varepsilon_n}(x)) \approx 2\varepsilon_n f_{sc,\theta}(x) + o_1(1)$ for a small error $o_1(1)$. Secondly, for a small error $o_2(1)$, if μ_{W_n} denotes the empirical spectral measure of W_n:

$$\mu_{(W_n,e_1)}(I_{\varepsilon_n}(x)) = \left(\frac{n}{|\mathcal{I}_{\varepsilon_n}(x)|} \sum_{i \in \mathcal{I}_{\varepsilon_n}(x)} |\langle \phi_i(W_n), e_1 \rangle|^2\right) \mu_{W_n}(I_{\varepsilon_n}(x)) \approx \left(\frac{n}{|\mathcal{I}_{\varepsilon_n}(x)|} \sum_{i \in \mathcal{I}_{\varepsilon_n}(x)} |\langle \phi_i(W_n), e_1 \rangle|^2\right) \times (2\varepsilon_n f_{sc,0}(x) + o_2(1)).$$

  Using local laws obtained by Knowles and Yin [1], one can prove that $o_1(1)$ and $o_2(1)$ are of smaller order than ε_n with high probability. Therefore, the left-hand side of (1) converges in probability towards the ratio of the densities $f_{sc,\theta}(x)/f_{sc,0}(x) = (\theta^2 - \theta x + 1)^{-1}$.

Related results

- **Generalizations.** The present results are taken from [4] and are particular instances of the study of the more general deformed models $W_n = n^{-1/2}X_n + A_n$ and $S_n = n^{-1}\Sigma_n^{1/2} X_n X_n^T \Sigma_n^{1/2}$, for general perturbations A_n and Σ_n whose normalized spectra respectively converge towards deterministic probability measures μ_A and μ_Σ. In these cases, the empirical spectral measures of W_n and S_n respectively converge towards the free convolution $\mu_{sc} \boxplus \mu_A$ and the free product $\mu_\alpha \boxtimes \mu_\Sigma$, where μ_{sc} is the semicircle law and μ_α the Marchenko-Pastur law with parameter α. If θ is an eigenvalue of A_n (resp. S_n) with associated eigenvector v_1, the spectral measures in direction v_1 still converge towards deterministic probability measures $\mu_{sc,A,\theta}$ and $\mu_{\alpha,\Sigma,\theta}$, whose Stieltjes transforms are explicit functions of the Stieltjes transforms of $\mu_{sc} \boxplus \mu_A$ and $\mu_\alpha \boxtimes \mu_\Sigma$. In these settings, the outliers of W_n and S_n still correspond to the atoms of the limiting spectral measures and we prove the convergence of overlaps between the eigenvectors of W_n (resp. S_n) and v_1, averaged on small scales, towards the ratio of the densities of $\mu_{sc,A,\theta}$ (resp. $\mu_{\alpha,\Sigma,\theta}$) and $\mu_{sc} \boxplus \mu_A$ (resp. $\mu_\alpha \boxtimes \mu_\Sigma$).

- **Other works.** Consider the general deformed models $W_n = n^{-1/2}X_n + A_n$ and $S_n = n^{-1}\Sigma_n^{1/2} X_n X_n^T \Sigma_n^{1/2}$ introduced above and suppose that θ is an eigenvalue of A_n (resp. Σ_n) with associated eigenvector v_1. When θ belongs to the support of the asymptotic spectrum of A_n (resp. Σ_n), the convergence of the overlaps between the eigenvectors of W_n (resp. S_n) and v_1 is a *microscopic* confirmation of the results of Allez and Bouchaud [2] (in the Wigner setting) and Ledoit and Péché [3] (in the Wishart setting), who derived the asymptotic behavior of the overlaps $|\langle \phi_\lambda, v_\gamma \rangle|^2$ by taking the average over eigenvectors ϕ_λ associated to eigenvalues of W_n (resp. S_n) belonging to a *macroscopic* part of $\mathrm{Supp}\,(\mu_{sc} \boxplus \mu_A)$ (resp. $\mathrm{Supp}\,(\mu_\alpha \boxtimes \mu_\Sigma)$) and the average over eigenvectors v_γ of A_n (resp. Σ_n) belonging to a *macroscopic* part of the asymptotic spectrum of A_n (resp. Σ_n). When θ does not belong to the support of the asymptotic spectrum of A_n (resp. S_n), such a macroscopic average is not available whereas the spectral measure approach still works.

References

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[2] Romain Allez and Jean-Philippe Bouchaud. Eigenvector dynamics under free addition. *Random Matrices Theory Appl.*, 3(3):1450010, 17, 2014.
[3] Olivier Ledoit and Sandrine Péché. Eigenvectors of some large sample covariance matrix ensembles. *Probab. Theory Related Fields*, 151(1-2):233–264, 2011.
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